1976

Lyapunov stability of large-scale dynamical systems

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76-18,289

RASMUSSEN, Robert Donald, 1949-
LYAPUNOV STABILITY OF LARGE-SCALE DYNAMICAL SYSTEMS.

Iowa State University, Ph.D., 1976
Engineering, system science

Xerox University Microfilms, Ann Arbor, Michigan 48106

Lyapunov stability of large-scale dynamical systems

by

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A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major: Electrical Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1976

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I. INTRODUCTION

In engineering the study of groups of interrelated components acting as a collective independent unit is usually referred to as systems theory. The concept of a system is quite general, however, and it is consequently appropriate also in such fields as the physical sciences, economics, ecology, sociology, and so on. Of interest is the modelling, simulation, and analysis of such systems, the goal being not only to understand them, but to predict and control their behavior as well.

Since the nature of most systems is dynamic, their history is of particular interest. An important method of modelling this evolution is to indicate the trend in the state of a system due to its immediate condition. If the state of a system has been quantified in some manner, this evolutionary model takes the form of a differential equation. At this point mathematics becomes an important tool to the systems theorist.

To achieve greater generality in such a differential equation model, the system state may be assumed to belong to some abstract space. The resulting theory is then applicable to a variety of specific cases. Since it may also be necessary to study systems subject to random influences, stochastic differential equations are also of interest. In this dissertation one aspect of systems theory is presented in the context of both abstract and stochastic models. This is the problem of large-scale systems stability.

Among the difficulties encountered with large-scale systems is that methods of analysis cannot generally be applied in a straightforward manner due to the size and complexity of such systems. This may in fact be used as a definition of the term "large-scale". Large-scale systems theory involves the development of procedures for applying existing theory in a manner which makes an analysis tractable. A significant approach to this problem has been to isolate various portions of a system so that the resulting subsystems are sufficiently small to permit analysis. Based on the discovered properties of the isolated subsystems and their interconnecting structure, one may often deduce properties of the so-called composite or interconnected system.

One property which has been successfully treated by this method is stability. The major concerns of stability analysis are to determine the sensitivity of the system state to perturbation, and also to discover any tendency toward some preferred or equilibrium state.

A particularly useful method of stability analysis, which will be of concern here, is the direct method of Lyapunov, by which one investigates a scalar measure of the deviation of the system state from equilibrium. No specific history of the system need be known since only the differential equation and an appropriate Lyapunov functional are required. The Lyapunov stability of large-scale systems has been considered by several investigators (see, e.g. [1]-[11]). The results presented herein represent a useful extension of many previous results in that they are formulated first in the setting of differential equations on Banach spaces, and second for a variety of stochastic differential equations. This includes a large number of cases not previously considered and allows, as an extra advantage, the analysis of hybrid systems (i.e., systems described by mixed types of differential equations). In addition, the theory of M-matrices has been applied to obtain several new stability theorems. In order to demonstrate the usefulness of these results several specific examples are included. Finally, in the last section these results will be compared to earlier results in more detail in order that they may be seen in their proper perspective.

II. NOTATION

In the subsequent development the following notation will be used. Specialized notation will be explained in its appropriate context.

Let R^n denote a Euclidean n-space, and let $|\cdot|$ denote the Euclidean norm. In particular, R will be the real numbers, and the nonnegative real numbers will be denoted $R^+ = [0,\infty)$. A vector in R^n is specified as $x = (x_i)$, $i = 1,\ldots,n$. Such a vector is said to be positive, i.e. $x > 0$, if $x_i > 0$, $i = 1,\ldots,\ell$. The transpose of x is denoted as x'.

Let an $m \times n$ matrix be denoted as $A = ((a_{ij}))$, $i = 1,\ldots,m$, $j = 1,\ldots,n$. The transpose of A is denoted as A'. For a square matrix B let $\lambda(B)$ be an eigenvalue of B, $\det(B)$ be the determinant of B, and $\mathrm{tr}(B)$ be the trace of B. If B is symmetric, $\lambda_M(B)$ and $\lambda_m(B)$ will represent the maximum and minimum eigenvalues of B, respectively. The matrix norm $\|A\|_m = [\mathrm{tr}(A'A)]^{1/2}$ will be used.

Let δ_{ij} denote the Kronecker delta, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Banach spaces will be denoted by X or Z with appropriate subscripts where necessary. Norms on X or Z are denoted by $\|\cdot\|$ with subscripts referring to the corresponding space.

The Banach space $X = L_p[a,b]$ will be used, where $L_p[a,b]$ is the space of Lebesgue measurable functions on the interval $[a,b]$ with norm $\|f\| = [\int_a^b |f|^p d\mu]^{1/p} < \infty$, $1 \leq p < \infty$.

Time derivatives are expressed by a dot over the variable (e.g., $\dot{x}$), while first and second order partial derivatives are given by $\nabla_x = (\partial/\partial x_i)$, $i = 1,\ldots,n$, and $\nabla_{xy} = ((\partial^2/\partial x_i \partial y_j))$, $i = 1,\ldots,n$, $j = 1,\ldots,m$, where $x \in R^n$, $y \in R^m$.

Comparison functions $\psi : R^+ \to R^+$ are said to be of class K (i.e., $\psi \in K$) if they are continuous, strictly increasing functions on R^+, and if $\psi(0) = 0$. If $\psi \in K$ and $\lim_{r\to\infty} \psi(r) = \infty$, then ψ is said to be of class KR (i.e., $\psi \in KR$). Two comparison functions ψ_1, $\psi_2 \in KR$ are said to be of the same order of magnitude if there exist positive constants k_1 and k_2 such that $k_1\psi_1(r) \leq \psi_2(r) \leq k_2\psi_1(r)$ for all $r \in R^+$. (For a discussion of comparison functions, see Hahn [12].)

Let $o(r)$ denote terms of second or higher order in r so that $\lim_{r\to 0} o(r)/r = 0$.

III. DYNAMICAL SYSTEMS ON BANACH SPACES

The continuing use of Lyapunov's direct method as an important tool in the qualitative analysis of dynamical systems has been augmented in recent years by the extension of this method to systems defined on abstract spaces. Such systems have been considered, e.g., in [13]-[20], and numerous other references. The stability of such systems has been considered in [21]-[30], and others.

Dynamical systems are often described by semigroups of transition operators which define explicitly the history of the system. Several types of dynamical systems are considered in the literature including strong [19]-[22], weak [23], extended [24], and limit [25] dynamical systems. In the following section, strong dynamical systems will be considered with indications of how the results may also be applied to weak dynamical systems.

<u>Definition 1</u> [30]. Let $\{T_t\}$, $t \in R^+$, be a family of mappings of a Banach space X into itself. Then T_t is a semi-group of transition operators defining a <u>strong dynamical system</u> on X with trajectories

$$x_t = T_t a \quad , \qquad x_0 = a \quad , \tag{1}$$

if

(i) $T_t a$ is continuous in both t and a,

(ii) $T_0 a = Ia = a$, I denoting the identity operator, and

(iii) $T_t T_\tau a = T_{t+\tau} a$,

for all $t, \tau \in R^+$ and $a \in X$.∎

It is assumed, henceforth, that

$$T_t 0 \equiv 0 \quad , \quad t \in R^+ \quad . \tag{2}$$

This will be referred to as the trivial solution.

It often occurs that the most one can determine for a dynamical system is weak continuity. In this case the system is called a weak dynamical system. It is still possible, however, to apply the theory to be presented, given certain modifications. If X is a separable and reflexive Banach space, then the weak topology on X is metrizable with some metric ρ. Replacing $\|x\|$ by $\rho(x,o)$ and replacing all topological and continuity properties with their weak analogs will lead to weak stability results which are similar to the strong stability results to be derived. In fact, in a number of instances, for example, finite dimensional systems, the weak and strong results are equivalent.

Although it is possible to express stability theorems for (1) (see e.g. [12], [22]), physical systems are rarely described in semigroup form. Rather, system trajectories are usually defined by the solutions of a differential equation on X of the form

$$\dot{x}_t = Ax_t \quad , \quad x_0 = a \quad . \tag{3}$$

The operator A, possibly nonlinear, is assumed to have its domain D(A) dense in X. It is also assumed that A0 = 0.

A function $x_t : R^+ \to X$ is said to be a solution of (3) if $x_t \in D(A)$ and possesses a derivative which satisfies (3) for all $t \in R^+$. Henceforth, system (3) is assumed to be <u>well-posed</u> in the sense that it possesses a unique solution for each $x_0 = a \in D(A)$, and solutions depend continuously on a for all $t \in R^+$.

Under the assumption of well-posedness, the solutions of (3) determine the semigroup T_t on D(A) with $T_t a \triangleq x_t$, $x_0 = a \in D(A)$. Since D(A) is dense in X, T_t may be extended continuously to X. Therefore, (3) defines a strong dynamical system on X. The operator A is known as the *strong infinitesimal generator* of T_t, since

$$Ax = \lim_{t \to 0^+} A_t x \quad , \quad A_t \triangleq t^{-1}(T_t - I) \quad , \tag{4}$$

for all $x \in D(A)$.

Consider now the following examples.

Example 1: Consider the n-dimensional linear case where $x_t \in R^n$ satisfies the equation

$$\dot{x}_t = Ax_t \quad , \quad x_0 = a \quad , \tag{5}$$

A being an $n \times n$ matrix. It is well known that the semigroup for (5) is the $n \times n$ transition matrix $\exp(At)$. The domain of A is all of R^n.

In the nonlinear case on R^n conditions for the existence, uniqueness and continuity of solutions are known also, but in general a semigroup solution is difficult or impossible in closed form.

The theory of linear differential equations has been extended to infinite dimensional spaces and unbounded linear operators with considerable success (see, e.g. [19], [20], [27], [30], and so on).

As an example of such a linear system consider the integro-differential equation

$$\frac{\partial}{\partial t} x_t(u) = \lambda x_t(u) + \int_a^b K(u,v)x_t(v)dv \tag{6}$$

where the kernel K is sufficiently smooth and $x_t(u) \in L_p[a,b]$ for each $t \in R^+$. Equation 6 is of the form (3) and it can be shown (see [21]) that, given $a = a(u)$, then

$$x_t(u) = T_t a(u) = e^{\lambda t}[a(u) + \int_o^t \int_a^b R(u,v,\tau)a(v)dvd\tau] \tag{7}$$

where $R(u,v,\tau)$ is the resolvent of $K(u,v)$.∎

A differential equation may not be in the proper form as it is in the example above. This is seen, for instance, in the following case.

<u>Example 2</u>: The functional differential equation

$$\frac{d}{dt} x(t) = g(x_t) \quad , \quad x_0(\tau) = a(\tau) \quad , \tag{8}$$

where $x(t) \varepsilon R$ for each $t \varepsilon R^+$ and $x_t(\tau) = x(t+\tau)$ for $\tau \varepsilon [-d,o]$, is not in the form of Equation 3 since the domain of the functional g is not R, but rather some function space.

However by setting

$$Ax_t(\tau) = \begin{cases} \frac{\partial}{\partial\tau} x_t(\tau) & , \quad \tau \varepsilon [-d,o) \\ g(x_t(\tau)) & , \quad \tau = 0^- \end{cases} \tag{9}$$

it may be shown that solutions of (8) and (3) are equivalent. Existence, uniqueness and continuity conditions for (8) may be found in [18], [30], and so on.∎

Numerous other examples of systems which are of the form (3) could be given, including certain classes of partial differential equations, systems of incomplete information, differential-difference equations, and others. For such examples see [12], [21], [22], [26], and so on.

In Chapter V it will be shown how the diffusion equation, a partial differential equation of the form (3), plays an important role in the stability analysis of stochastic systems.

The Lyapunov stability of system (3) is defined in the usual manner as follows.

Definition 2: The trivial solution is said to be asymptotically stable if, given $x_0 = a$,

(i) for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $a \in X$, $\|a\| < \delta$ implies $\|x_t\| < \varepsilon$ for all $t \in R^+$, and

(ii) there is a $\delta_m > 0$ such that for any $a \in X$, $\|a\| < \delta_m$ implies $\|x_t\| \to 0$ as $t \to \infty$. ∎

Definition 3: The trivial solution is said to be exponentially stable if, given $x_0 = a$, condition (i) of Definition 2 is satisfied and

(ii)' there are positive constants δ_m, M and β such that for any $a \in X$, $\|a\| < \delta_m$ implies $\|x_t\| < Me^{-\beta t}$ for all $t \in R^+$. ∎

When the δ_m's in the above definitions can be made arbitrarily large, the respective stabilities are said to be global.

Stability results for (3) will be expressed in terms of scalar functions $V : X \to R$ and the upper right Dini derivative of V along solutions of (3) given as

$$\dot{V}(a) \triangleq \overline{\lim_{t \to 0^+}} \; t^{-1}[V(x_t) - V(a)] \quad . \tag{10}$$

The ordinary derivative is not used since it may be undefined.

In order to take advantage of (3) in the expression for $\dot{V}$, it is necessary to restrict the class of admissible V functionals.

<u>Definition 4</u>: A function $V : X \to R$ is said to be an <u>admissible Lyapunov functional</u> if

(i) $V(0) = 0$,

(ii) V is continuous on X,

(iii) the closure of the set

$$Q_m = \{a \in X : V(a) < m\} , \tag{11}$$

denoted by $\overline{Q}_m$, is bounded for all $m > 0$, and

(iv) there exists a function $\nabla V : X \times X \to R$ such that for all $a, x \in X$,

(a) $V(a+x) - V(a) \leq \nabla V(a,x) + o(\|x\|)$

(b) $\nabla V(a,x)$ is linear and continuous in x, uniformly with respect to $a \in Q_m$ for each $m > 0$. ∎

Under the assumptions of Definition 4 and given (1) and (4), one has for all $a \varepsilon \overline{Q}_m \cap D(A)$ and any $m > 0$,

$$\begin{aligned}\dot{V}(a) &= \overline{\lim_{t\to o^+}}\ t^{-1}[V(T_t a) - V(a)] \\ &\leq \overline{\lim_{t\to o^+}}\ t^{-1}[\nabla V(a, T_t a - a) + o(\|T_t a - a\|)] \\ &\leq \overline{\lim_{t\to o^+}}\ \nabla V(a, A_t a) + \overline{\lim_{t\to o^+}}\ t^{-1} o(t\,\|A_t a\|) \\ &= \nabla V(a, Aa) \quad . \qquad (12)\end{aligned}$$

It follows from the fact that D(A) is dense and from the continuity of V and the solutions x_t that, given $\nabla V(a,Aa) < 0$ for $a \varepsilon \overline{Q}_m \cap D(A)$, then $\dot{V}(a) < 0$ for all $a \varepsilon \overline{Q}_m$. On this basis the following stability theorems may be stated.

<u>Theorem 1</u>: Given the assumptions for (3), suppose there exist an admissible Lyapunov functional V on X and three functions ψ_1, $\psi_2 \varepsilon KR$, $\psi_3 \varepsilon K$, such that for some $m > 0$,

(i) $\psi_1(\|a\|) \leq V(a) \leq \psi_2(\|a\|)$, $a \varepsilon \overline{Q}_m$, and

(ii) $\nabla V(a,Aa) \leq - \psi_3(\|a\|)$, $a \varepsilon \overline{Q}_m \cap D(A)$.

Then the trivial solution of (3) is <u>asymptotically stable</u>.∎

<u>Theorem 2</u>: If in Theorem 1 the functions ψ_1, ψ_2 and ψ_3 are of the same order of magnitude in class KR, then the trivial solution of (3) is <u>exponentially stable</u>.∎

These theorems, which are readily shown (see, e.g. Hahn [12]), are the results on which the theorems of the next chapter are based. If the hypotheses of Theorems 1 and 2 can be shown to hold for an arbitrarily large m, then one is able to conclude that the respective stabilities are global.

IV. MAIN RESULTS: LARGE-SCALE SYSTEMS DEFINED ON BANACH SPACES

Large-scale systems will now be considered which are in the form of interconnected subsystems. The isolated subsystems $\mathcal{S}_i$, $i=1,\ldots,\ell$, are introduced first. Then, it is shown how these subsystems are to be interconnected to form the composite system $\mathcal{S}$. The remainder of this chapter contains the main results for asymptotic and exponential stability of $\mathcal{S}$. The proofs of these theorems may be found in Appendix A.

The <u>isolated subsystems</u> are differential equations in the manner of (3). That is,

$$\mathcal{S}_i \; : \; \dot{z}_t^i = F_i z_t^i \quad , \quad z_0^i = a_i \quad , \quad i=1,\ldots,\ell \tag{13}$$

are assumed to satisfy all restrictions satisfied by (3), where $z_t^i \in Z_i$, Z_i being the subsystem state Banach space with norm $\|\cdot\|_i$.

By choosing the subsystems $\mathcal{S}_i$ of sufficiently low order each may be analyzed by Lyapunov's direct method as outlined in the previous chapter. The resulting information is summarized in the following properties.

<u>Definition 5</u>: An isolated subsystem $\mathcal{S}_i$ is said to possess <u>Property A</u> if there exist an admissible Lyapunov functional V_i, three functions ψ_{i1}, $\psi_{i2} \in KR$, $\psi_{i3} \in K$, and a real constant

σ_i such that for some $m_i > 0$

$$\text{(i)}\ \psi_{i1}(\|a_i\|_i) \leq V_i(a_i) \leq \psi_{i2}(\|a_i\|_i)\ ,\ a_i \in \overline{Q}_{m_i},$$

and

$$\text{(ii)}\ \nabla V_i(a_i, F_i a_i) \leq \sigma_i \psi_{i3}(\|a_i\|_i)\ ,\ a_i \in \overline{Q}_{m_i} \cap D(F_i)\ .\blacksquare$$

<u>Definition 6</u>: If in Definition 5 the functions ψ_1, ψ_2 and ψ_3 are of the same order of magnitude in class KR, then isolated subsystem $\mathcal{S}_i$ is said to possess <u>Property B</u>.$\blacksquare$

Clearly, if $\sigma_i < 0$, then Definitions 5 and 6 correspond to the hypotheses of stability Theorems 1 and 2. The constant σ_i may be loosely interpreted as a <u>damping factor</u>, and it is therefore a measure of the degree of stability of the subsystem $\mathcal{S}_i$. This will be useful in studying the effects of subsystems on the behavior of the entire interconnected system.

The composite system state consists of the vector of subsystem states, $x = (z^i)$, $i=1,\ldots,\ell$. Letting

$$X = \mathop{\mathsf{X}}_{i=1}^{\ell} Z_i$$

be the composite system space, define the composite norm

$$\|x\| = \max_i \|z^i\|_i \quad . \tag{14}$$

Under this norm X is a Banach space.

The subsystems are interconnected in the following fashion to comprise the composite system

$$\mathcal{S}: \dot{z}_t^i = F_i z_t^i + G_i x_t \quad , \quad i=1,\ldots,\ell \tag{15}$$

with $x_0 = a = (a_i)$, $i=1,\ldots,\ell$. The operator G_i, defining the interconnection structure, has domain $D(G_i) \subset X$ and maps $D(G_i)$ into Z_i.

It should be noted that, although the additive nature of the interconnections in (15) may appear to be restrictive, it is, nevertheless, always possible to achieve such a decomposition. It may occur that, either by choice or by necessity, one or more of the operators F_i is zero. However, this case is not excluded by Properties A or B. The introduction of the isolated subsystems $\mathcal{S}_i$ is therefore as much a conceptual tool as it is a natural formulation of the problem.

Letting $Ax = (F_i z^i + G_i x)$, $i=1,\ldots,\ell$, Equation 15 can be expressed equivalently as

$$\mathcal{S}: \dot{x}_t = Ax_t \quad , \quad x_0 = a \; , \tag{16}$$

which is clearly identical to (3). It will be assumed that all restrictions on (3) hold as well for (16). Therefore Theorems 1 and 2 may also be applied to (16) and this will be the basis for subsequent results.

The proofs of the theorems in this section may be found in Appendix A.

Theorem 3: Assume that composite system $\mathcal{S}$ satisfies the following conditions:

(i) each isolated subsystem $\mathcal{S}_i$ possesses Property A;

(ii) given the Lyapunov functionals V_i and comparison functions ψ_{i3}, $i=1,\ldots,\ell$, of hypothesis (i), there exist real constants b_{ij}, $i,j=1,\ldots,\ell$, such that

$$\nabla V_i(a_i, G_i a) \leq [\psi_{i3}(\|a_i\|_i)]^{1/2} \sum_{j=1}^{\ell} b_{ij}[\psi_{j3}(\|a_j\|_j)]^{1/2} \tag{17}$$

for all $a \in \mathop{\times}\limits_{i=1}^{\ell} Q_{m_i} \cap D(A)$; and

(iii) there exist positive constants α_i, $i=1,\ldots,\ell$, such that the test matrix $S = ((s_{ij}))$, $i,j=1,\ldots,\ell$, defined by

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + b_{ii}), & i=j \\ \frac{1}{2}(\alpha_i b_{ij} + \alpha_j b_{ji}), & i \neq j \end{cases} \tag{18}$$

is negative definite.

Then the trivial solution of $\mathcal{S}$ is asymptotically stable.∎

<u>Theorem 4</u>: Assume that composite system $\mathcal{S}$ satisfies the following conditions:

(i) each isolated subsystem possesses Property B and all comparison functions ψ_{ij}, $i=1,\ldots,\ell$, $j=1,2,3$, are of the same order of magnitude;

(ii) hypotheses (ii) and (iii) of Theorem 3 hold.

Then the trivial solution of $\mathcal{S}$ is <u>exponentially stable</u>.∎

It is emphasized that the above results express the stability of $\mathcal{S}$ in terms of the lower order subsystem properties, and in terms of bounds on the interconnecting structure (Equation 17). The confining relationship between these properties is determined by the test matrix S.

The matrix S is of particular interest since a number of observations may be made regarding the negative definiteness condition.

First, note that a necessary condition for negative definiteness is

$$\sigma_i + b_{ii} < 0 \, , \quad i=1,\ldots,\ell \quad . \tag{19}$$

Thus, each subsystem must either possess a certain degree of stability, or the interconnecting structure must provide local stabilizing feedback around unstable subsystems.

Second, note that the nature of the bounds on the interconnecting structure is to express their strength, relative to the subsystem damping. The negative definiteness condition has the effect of limiting the degree to which the interconnecting structure effects the behavior of the composite system. Therefore, these results are essentially weak coupling conditions. That is, given the degree of stability of the subsystems (with the local feedback), the condition on S determines a permissible strength of interconnection below which one may conclude stability for the interconnected system. This appears to be characteristic of most results obtained so far for large-scale systems.

The observations just cited suggest a systematic procedure for the stabilization of unstable large-scale systems through the use of local stabilizing feedback around the subsystems. With subsystems of sufficiently low order existing stabilization techniques could be applied with relative ease. The degree of stabilization needed would be determined by the condition on the test matrix S.

Note that in using Theorems 3 and 4 it is necessary to find the positive constants α_i, $i=1,\ldots,\ell$, such that S will be negative definite. It is not evident in advance that such constants exist, and although the choice of such constants is not unique, one may not be fortunate in finding an appropriate set. If the constants b_{ij}, $i\neq j$, $i,j=1,\ldots,\ell$, are nonnegative, the necessity of choosing arbitrary constants may be eliminated. This will be accomplished in the following results. It should be noted, however, that no such restriction on the sign of the b_{ij}'s was made in Theorems 3 and 4. Therefore these theorems will remain important due to their greater generality.

The assumption of nonnegativeness on the sign of the off-diagonal terms in (18) is useful in that it permits the use of the theory of M-matrices in expressing stability conditions for $\mathcal{S}$.

<u>Definition 7</u> [4]: A matrix $D = ((d_{ij}))$, $i,j=1,\dots,\ell$, is said to be an M-matrix if $d_{ij} \leq 0$ for all $i \neq j$, and if one of the following equivalent conditions is satisfied:

(i) the successive principal minors of D are each positive;

(ii) there is a vector $x > 0$ such that $Dx > 0$;

(iii) there is a vector $y > 0$ such that $D'y > 0$;

(iv) D is nonsingular and all elements of D^{-1} are nonnegative; and

(v) the real parts of the eigenvalues of D are all positive.∎

If D is an M-matrix, the existence of a diagonal matrix W with positive diagonal elements can be shown such that $WD + D'W$ is positive definite (see Appendix C). This and a number of other properties of M-matrices can be found in [4], [31], [32], and so on. These results are used to yield the following theorems.

Theorem 5: Assume that conditions (i) and (ii) of Theorem 3 hold with $b_{ij} \geq 0$ for all $i \neq j$. If the successive principal minors of the test matrix $D = ((d_{ij}))$, $i,j=1,\ldots,\ell$, are positive, where

$$d_{ij} = \begin{cases} -(\sigma_i + b_{ii}) , & i=j \\ -b_{ij} , & i \neq j \end{cases} , \tag{20}$$

then the trivial solution of $\mathcal{S}$ is asymptotically stable.∎

Note that this test matrix condition is completely computational, involving no arbitrary constants to be chosen. Thus, Theorem 5 offers a distinct advantage over Theorem 3. The next theorem, while reintroducing arbitrary constants, is in a form which illustrates very clearly the weak coupling nature of these results.

Theorem 6: Assume that conditions (i) and (ii) of Theorem 3 hold with $b_{ij} \geq 0$ for all $i \neq j$. If there exist positive constants λ_i, $i=1,\ldots,\ell$, such that

$$(\sigma_i + b_{ii}) < - \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \left(\frac{\lambda_j}{\lambda_i}\right) b_{ij} \leq 0 , \quad i=1,\ldots,\ell, \tag{21}$$

then the trivial solution of $\mathcal{S}$ is asymptotically stable.∎

One can now see immediately that Equation 19 is satisfied, and that the strength of the interconnections between subsystems must be limited as indicated previously.

The choice of constants in Theorem 6 is less difficult than in Theorem 3 since the condition to be satisfied is simpler. Linear programming techniques appear to be appropriate in this case.

It is possible to obtain other theorems which are similar to Theorems 5 and 6 by using the theory of M-matrices. For example, the condition on the text matrix D in Theorem 5 could be changed to require that D must possess a positive inverse or that D must have its eigenvalues in the right half plane. This is made evident in Appendix A where the previous two theorems are shown to be mathematically equivalent. These two were chosen since they appear to be the most useful among the alternatives. An investigator can therefore choose among several possible tests for the stability of large-scale systems.

It can be shown that if a matrix D is an M-matrix, then $D-\mu I$ is an M-matrix if and only if $\mu < \min \mathrm{Re}[\lambda(D)]$ (see Appendix C). This is the basis for the following important result.

<u>Theorem 7</u>: Let the matrix D be defined as in Theorem 5, and assume that composite system $\mathcal{S}$ has been shown to be asymptotically stable by either Theorem 5 or 6. Then any modification of the subsystems $\mathcal{S}_i$ (or their local feedback)

which increases each $(\sigma_i + b_{ii})$ by less than $\mu = \min \mathrm{Re}[\lambda(D)]$ will leave the system $\mathcal{S}$ asymptotically stable.∎

In this sense μ may be interpreted as a <u>margin of stability</u> for the large-scale system. It may be used to judge how sensitive the stability is with respect to structural changes and is therefore a useful parameter.

Theorems 5-7 can readily be extended to include exponential as well as asymptotic stability. This is accomplished in the manner of Theorem 4 by simply requiring all comparison functions to be of the same order of magnitude.

The following examples will serve to illustrate the manner in which the previous theorems are applied. In the first example a simple hybrid system is given as a demonstration that such systems may be approached by the method presented. The second more complex example represents an actual system arising from the field of nuclear reactor dynamics.

<u>Example 3</u>: Consider the <u>hybrid system</u> described by the equations

$$\left.\begin{aligned} \dot{z}_t^1 &= Az_t^1 + b\int_0^L f(y)z_t^2(y)dy \\ \dot{z}_t^2 &= \alpha\nabla_{yy}z_t^2(y) - \beta z_t^2(y) + g(y)c'z_t^1 \end{aligned}\right\} \qquad (22)$$

where $z_t^1 \in R^n$ and $z_t^2 \in L_2[0,L]$. The second subsystem state z_t^2 is assumed to satisfy the boundary condition

$$\nabla_y z_t^2(y)^2 \Big|_0^L \leq 0 \quad , \quad t \in R^+ \quad . \qquad (23)$$

In addition, A is assumed to be a stable $n \times n$ matrix. The constants α and β are positive, b and c are n vectors, and f, g $\in L_2[0,L]$ are sufficiently smooth to guarantee solutions for (22).

The isolated subsystems are chosen as

$$\left.\begin{aligned} \mathcal{S}_1 &: \dot{z}_t^1 = Az_t^1 \quad , \quad \text{and} \\ \mathcal{S}_2 &: \dot{z}_t^2 = \alpha\nabla_{yy}z_t^2(y) - \beta z_t^2(y) \end{aligned}\right\} \qquad (24)$$

Since A is stable, there exists a positive definite symmetric matrix P such that $A'P + PA \triangleq -Q$ is negative definite. Choosing $V_1(a_1) = a_1{}'Pa_1$, one has

$$\psi_{11}(|a_1|) \triangleq \lambda_m(P)|a_1|^2 \leq V_1(a_1) \leq \lambda_M(P)|a_1|^2 \triangleq \psi_{12}(|a_1|) \ , \qquad (25)$$

and

$$\nabla V_1(a_1, F_1 a_1) = \nabla_{a_1} V_1(a_1)' F_1 a_1 = -a_1' Q a_1 \leq -\lambda_m(Q)|a_1|^2$$

$$= \sigma_1 \psi_{13}(|a_1|) \tag{26}$$

where

$$\psi_{13}(r) = r^2$$

and

$$\sigma_1 = -\lambda_m(Q) \quad . \tag{27}$$

For $\mathcal{S}_2$ choose $V_2(a_2) = \frac{1}{2}\|a_2\|_2^2$ so that

$$\psi_{21}(\|a_2\|_2) \triangleq V_2(a_2) \triangleq \psi_{22}(\|a_2\|_2) \quad , \tag{28}$$

and

$$\nabla V_2(a_2, F_2 a_2) = \frac{1}{2}\int_0^L \nabla_{a_2} a_2(y)^2 F_2 a_2(y) dy$$

$$= \int_0^L a_2(y)[\alpha \nabla_{yy} a_2(y) - \beta a_2(y)] dy$$

$$= \int_0^L [-\alpha(\nabla_y a_2(y))^2 - \beta a_2(y)^2] dy + \frac{\alpha}{2} \nabla_y a_2(y)^2 \Big|_0^L$$

$$\leq -\beta \|a_2\|^2 = \sigma_2 \psi_{23}(\|a_2\|) \tag{29}$$

where

$$\psi_{23}(r) = r^2$$

and

$$\sigma_2 = -\beta \quad . \tag{30}$$

Clearly, isolated subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ both possess Property B.

Next consider the bounds on the interconnecting structure. One has

$$\nabla V_1(a_1, G_1 a) = \nabla_{a_1} V(a_1)' G_1 a = 2a_1' Pb \int_0^L f(y) a_2(y) dy$$

$$\leq 2|a_1| \lambda_m(P) |b| \; \|f\|_2 \; \|a_2\|_2 \tag{31}$$

which implies that

$$b_{11} = 0 \quad , \quad b_{12} = 2 \; \lambda_m(P) \; |b| \; \|f\|_2 \quad . \tag{32}$$

Finally,

$$\nabla V_2(a_2, G_2 a) = \frac{1}{2} \int_0^L \nabla_{a_2} a_2(y)^2 G_2 a \; dy$$

$$= \int_0^L a_2(y) g(y) c' a_1 dy \leq \|a_2\|_2 \; \|g\|_2 \; |c| \; |a_1| \tag{33}$$

which implies that

$$b_{21} = \|g\|_2 \ |c| \quad , \quad b_{22} = 0 \quad . \tag{34}$$

The test matrix for Theorem 4 is therefore

$$S = \begin{bmatrix} -\alpha_1\lambda_m(q) & \alpha_2\lambda_M(P)|b| \ \|f\|_2 + \frac{1}{2}\alpha_1 \ \|g\|_2 |c| \\ \frac{1}{2}\alpha_1 \ \|g\|_2 \ |c| + \alpha_2\lambda_M(P)|b| \ \|f\|_2 & -\alpha_2\beta \end{bmatrix} \tag{35}$$

By choosing

$$\alpha_1 = 1/\lambda_M(P)|b| \ \|f\|_2 \quad , \quad \alpha_2 = 2/\|g\|_2 |c| \tag{36}$$

one finds that S is negative definite if and only if

$$\beta \left[\frac{\lambda_m(Q)}{\lambda_M(P)}\right] > 2 \ |b| \ |c| \ \|f\|_2 \ \|g\|_2 \tag{37}$$

That is, given the initial assumptions, hybrid system (22) satisfies the hypotheses of Theorem 4 and is therefore exponentially stable if inequality (37) is satisfied. The weak coupling nature of this condition is obvious.∎

Example 4: Consider the point kinetics model of a coupled core nuclear reactor with ℓ cores (see [33], [34]) described by the set of equations

$$\Lambda_i \dot{p}_i(t) = [\rho_i(t) - \varepsilon_i - \beta_i]p_i(t) + \rho_i(t) + \sum_{k=1}^{6} \beta_{ki}c_{ki}(t)$$
$$+ \sum_{j=1}^{\ell} \varepsilon_{ji} \frac{P_{j0}}{P_{i0}} \int_{-\infty}^{t} h_{ji}(t-s)p_j(s)ds \quad ,$$

$$\dot{c}_{ki}(t) = \lambda_{ki}[p_i(t) - c_{ki}(t)] \quad , \quad i=1,\ldots,\ell, \; k=1,\ldots,6 \tag{38}$$

where $p_i : R \to R$ and $c_{ki} : R \to R$ represent the power in the i^{th} core and the concentration of the k^{th} precursor in the i^{th} core, respectively. The constants Λ_i, ε_i, β_i, β_{ki}, ε_{ji}, P_{io}, and λ_{ki} are all positive, where

$$\beta_i = \sum_{k=1}^{6} \beta_{ki} \quad .$$

The functions $h_{ji} : R^+ \to R$ determine the coupling via neutron migration from the j^{th} core to the i^{th} core. The reactivity $\rho_i(t)$ of the i^{th} core is expressed by the relation

$$\rho_i(t) = \int_{-\infty}^{t} w_i(t-s)p_i(s)ds \quad , \quad i=1,\ldots,\ell \quad , \tag{39}$$

where $w_i : R^+ \to R$. Making the physically realistic assumption that

$$\lim_{t\to-\infty} c_{ki}(t)e^{\lambda_{ki}t} = 0 \quad , \quad k=1,\ldots,6 \; , \; i=1,\ldots,\ell \tag{40}$$

one obtains from (38)

$$c_{ki}(t) = \int_{-\infty}^{t} \lambda_{ki}e^{-\lambda_{ki}(t-s)} p_i(s)ds \quad , \quad k=1,\ldots,6 \; , \quad i=1,\ldots,\ell \; . \tag{41}$$

By using (39) and (41) to eliminate $\rho_i(t)$ and $c_{ki}(t)$ from (38), one is left with ℓ functional differential equations which will now be modified, as in Example 2, to be of the form (15).

In order to simplify the notation, let

$$f_i(t) \triangleq \Lambda_i^{-1}[w_i(t) + \sum_{k=1}^{6} \beta_{ki}\lambda_{ki}e^{-\lambda_{ki}t} + \varepsilon_{jj}h_{jj}(t)] \ , \tag{42}$$

$$K_i \triangleq \Lambda_i^{-1} (\varepsilon_j + \beta_j) \ , \tag{43}$$

$$n_i(t) \triangleq \Lambda_i^{-1}w_i(t) \ , \quad i=1,\ldots,\ell \ , \tag{44}$$

and

$$g_{ij}(t) \triangleq \frac{\varepsilon_{ji}P_{j0}}{\Lambda_i P_{i0}} h_{ji}(t) \ , \quad i\neq j \ , \ i,j=1,\ldots,\ell \ . \tag{45}$$

To put (38) into appropriate form, let

$$z_t^i(\tau) = p_i(t+\tau) \ , \ \tau \leq 0 \ , \ i=1,\ldots,\ell \ . \tag{46}$$

Eliminating $\rho_i(t)$ and $c_{ki}(t)$, making the change of variables $s = t+\tau$, and noting that $p_i(t) = z_t^i(0)$ one obtains $\dot{x}_t = Ax_t$, where $x_t = (z_t^i)$, $i=1,\ldots,\ell$, and

$$\dot{z}_t^i(\tau) = A_i x_t(\tau) = \begin{cases} -K_i z_t^i(0) + \int_{-\infty}^{0} f_i(-u) z_t^i(u)du \\ + z_t^i(0) \int_{-\infty}^{0} n_i(-u) z_t^i(u)du \\ + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{-\infty}^{0} g_{ij}(-u) z_t^i(u)du \ , \ \tau=0 \\ \nabla_u z_t^i(u)\Big|_{u=\tau} \ , \ \tau < 0 \ . \end{cases} \tag{47}$$

The initial condition is therefore $z_0^i(\tau) = a_i(\tau) = p_i(\tau)$, $\tau \leq 0$, $i=1,\ldots,\ell$, the past history of $p_i(t)$ at $t=0$.

For some $L_i > 0$, define the functions

$$m_i(u) = \begin{cases} L_i^{-1} e^{L_i u} \ , & u < 0 \\ L_i^{-1} + 1 \ , & u = 0 \end{cases} \qquad i=1,\ldots,\ell \tag{48}$$

and let Z_i, $i=1,\ldots,\ell$, be function spaces on $(-\infty,0]$ with their respective norms defined, using the Lebesgue-Stieltjes integral, as

$$\|z^i\|_i = \left[\int_{-\infty}^{0} z^i(u)^2 dm_i(u)\right]^{1/2}$$

$$= [z^i(0)^2 + \int_{-\infty}^{0} z^i(u)^2 e^{L_i u} du]^{1/2} , \quad i=1,\ldots,\ell \; . \tag{49}$$

Then Z_i is an L_2-space with the above norm and $z_t^i \varepsilon Z_i$.

System (47) may be viewed as an interconnection of ℓ isolated subsystems $\mathcal{S}_i$ described by

$$\dot{z}_t^i(\tau) = F_i z_t^i(\tau) = \begin{cases} -K_i z_t^i(0) + \int_{-\infty}^{0} f_i(-u) z_t^i(u) du \\ \quad + z_t^i(0) \int_{-\infty}^{0} n_i(-u) z_t^i(u) du , & \tau = 0 \\ \nabla_u z_t^i(u) \Big|_{u=\tau} , & \tau < 0 \end{cases} \tag{50}$$

$i=1,\ldots,\ell$. For $\mathcal{S}_i$ choose the Lyapunov functional

$$V_i(a_i) = a_i(0)^2 + K_i \int_{-\infty}^{0} a_i(u)^2 e^{L_i u} du , \quad i=1,\ldots,\ell \; . \tag{51}$$

Then

$$\min(1,K_i) \, \|a_i\|_i^2 \le V_i(a_i) \le \max(1,K_i) \, \|a_i\|_i^2 \tag{52}$$

and

$$\nabla V_i(a_i, F_i a_i) = 2a_i(0)F_i a_i(0)$$

$$+ 2K_i \int_{-\infty}^{0} a^i(u)F_i a_i(u)e^{L_i u}du \; . \qquad (53)$$

For initial conditions satisfying

$$\lim_{t\to-\infty} a_i(\tau)^2 e^{L_i\tau} = 0 \quad , \quad i=1,\ldots,\ell \quad , \qquad (54)$$

integration by parts yields

$$\int_{-\infty}^{0} a_i(u)F_i a_i(u)e^{L_i u}du = \int_{-\infty}^{0} a_i(u)\nabla_u a_i(u)e^{L_i u}du$$

$$= \frac{1}{2}a_i(0)^2 - \frac{1}{2}L_i \int_{-\infty}^{0} a_i(u)^2 e^{L_i u}du \; , \; i=1,\ldots,\ell \; . \qquad (55)$$

Let

$$b_i \triangleq \left[\int_{-\infty}^{0} a_i(u)^2 e^{L_i u}du\right]^{1/2} , \; i=1,\ldots,\ell \quad , \qquad (56)$$

and assume $L_i > 0$ can be chosen such that

$$c_i \triangleq \left[\int_{0}^{\infty} f_i(u)^2 e^{L_i u}du\right]^{1/2} \qquad (57)$$

and

$$d_i \triangleq \left[\int_0^{\infty} n_i(u)^2 e^{L_i u} du\right]^{1/2}, \quad i=1,\ldots,\ell \quad , \tag{58}$$

are finite. Applying (55) and the Cauchy-Schwarz inequality to (53), one obtains

$$\begin{aligned}
\nabla V_i(a_i, F_i a_i) &= 2a_i(0)\Big[-K_i a_i(0) + \int_{-\infty}^{0} f_i(-u) a_i(u) du \\
&\quad + a_i(0) \int_{-\infty}^{0} n_i(-u) a_i(u) du\Big] + 2K_i\Big[\tfrac{1}{2} a_i(0)^2 \\
&\quad - \tfrac{1}{2} L_i \int_{-\infty}^{0} a_i(u)^2 e^{L_i u} du\Big] \\
&= -K_i a_i(0)^2 + 2a_i(0) \int_{-\infty}^{0} [f_i(-u) e^{-L_i u/2}][a_i(u) e^{L_i u/2}] du \\
&\quad + 2a_i(0)^2 \int_{-\infty}^{0} [n_i(-u) e^{-L_i u/2}][a_i(u) e^{L_i u/2}] du \\
&\quad - K_i L_i \int_{-\infty}^{0} a_i(u)^2 e^{L_i u} du \\
&\leq -K_i a_i(0)^2 + 2c_i a_i(0) b_i - K_i L_i b_i^2 + 2d_i a_i(0)^2 b_i
\end{aligned} \tag{59}$$

$i=1,\ldots,\ell$, a polynomial in $a_i(0)$ and b_i. The first three terms are a quadratic form which is negative definite if

$$K_i \sqrt{L_i} > c_i \ , \ i=1,\ldots,\ell \ . \tag{60}$$

This may be interpreted as a condition which requires the most recent history of the reactor to dominate the dynamic behavior of the system. The fourth term in (59) is a third order term and therefore, if (60) is satisfied, then $\nabla V_i(a_i, F_i a_i)$ is negative definite in some neighborhood of the origin. That is, for any σ_i such that

$$0 > \sigma_i \geq -\frac{1}{2} K_i(L_i+1) + [\tfrac{1}{4} K_i(L_i-1)^2 + c_i^2]^{\frac{1}{2}} \tag{61}$$

(where the lower bound is the maximum eigenvalue of the quadratic terms) one has

$$\nabla V_i(a_i, F_i a_i) \leq \sigma_i(a_i(0)^2 + b_i^2) = \sigma_i \, \|a_i\|_i^2 \ , \ i=1,\ldots,\ell \tag{62}$$

for $\|a_i\|_i$ sufficiently small. It follows from (52) and (62) that subsystems $\mathcal{S}_i$ possess Property B.

For the interconnections, let

$$c_{ij} = [\int_0^\infty g_{ij}(u)^2 e^{L_j u} du]^{\frac{1}{2}} \ , \ i \neq j \ , \ i,j=1,\ldots,\ell \ . \tag{63}$$

Then

$$\nabla V_i(a_i, G_i a) = 2a_i(0) \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{-\infty}^{0} g_{ij}(-u)a_j(u)du$$

$$= 2a_i(0) \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{-\infty}^{0} [g_{ij}(-u)e^{-L_j u/2}][a_j(u)e^{L_j u/2}]du$$

$$\leq a_i(0) \sum_{\substack{j=1 \\ j\neq i}}^{\ell} 2c_{ij}b_j \leq \|a_i\|_i \sum_{\substack{j=1 \\ j\neq i}}^{\ell} 2c_{ij} \ \|a_j\|_j \quad . \quad (64)$$

Since $\psi_{j3}(r) = r^2$ it follows that

$$b_{ii} = 0 \ , \ b_{ij} = 2c_{ij} > 0 \ , \ i\neq j, \ i,j=1,\ldots,\ell \quad . \qquad (65)$$

Theorems 5-7 may now be applied given the constants in (61) and (65). For example, if $\ell = 2$ Theorem 5 yields the condition $\sigma_1\sigma_2 > 4c_{12}c_{21}$.

Thus, it may be concluded that, if each core of a coupled core nuclear reactor is exponentially stable when isolated, and if the coupling between cores via neutron migration is sufficiently weak (as determined by Theorems 5 or 6), then the reactor is also exponentially stable.

Finally, note that if $d_i = 0$, $i=1,\ldots,\ell$, then the system equations (38) are linear and the third order term in (59) is zero. Consequently, the lower bound on σ_i in (61) may be utilized in (62) for arbitrarily large $\|a_i\|_i$. Therefore, in this case the exponential stability would be global. ∎

V. STOCHASTIC DYNAMICAL SYSTEMS

In addition to dynamical systems on Banach spaces, Lyapunov's direct method has also been applied in recent years to stochastic systems whose trajectories are Markov processes. Systems of interest, however, are such that trajectories are not solutions of a differential equation in the same sense as discussed in Chapter III. The procedure which enables the Lyapunov stability analysis of stochastic systems is to utilize the dynamics of the so-called backward diffusion equation for Markov processes (see, e.g. [35], [36]). The resulting stability theorems, while similar to those given in Chapter III, must be considered independently of the previous results due to second order effects which can occur.

Random processes $x_t(\omega) \varepsilon R^n$, $t \varepsilon R^+$, will be considered which are defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, where Ω is the event space, $\mathcal{A}$ is a σ-algebra of events in Ω, and $\mathcal{P}$ is a probability measure on $\mathcal{A}$. The random behavior of x_t is characterized by the <u>distribution function</u>

$$P(t,B) = \mathcal{P}\{x_t \varepsilon B\} \quad , \tag{66}$$

and the <u>transition function</u>

$$P_a(t,B) = \mathcal{P}\{x_{t+\tau} \varepsilon B | x_\tau = a\} \quad , \tag{67}$$

the latter denoting a conditional probability. The evolution of the distribution function is completely determined by the transition function as follows

$$P(t+\tau,B) = \int_{a\varepsilon R^n} P_a(t,B)P(\tau,da) \quad . \tag{68}$$

Define the operators T_t, $t\varepsilon R^+$, on the functionals of R^n as the conditional expectation

$$v_t(a) = T_tV(a) \triangleq E_aV(x_t) = \int_{b\varepsilon R^n} V(b)P_a(t,db) \quad . \tag{69}$$

Then, if x_t is a homogeneous Markov process, it can be shown that T_t is a semigroup and can therefore define a dynamical system as in Chapter III. Letting A be the infinitesimal generator of T_t (which exists, for example, if x_t is right continuous), one obtains

$$\dot{v}_t = Av_t \ , \quad v_0(a) = V(a) \quad . \tag{70}$$

This is known as the <u>backward diffusion equation</u> of x_t.

In particular, if

$$V(a) = I_B(a) = \begin{cases} 1, & a\,\varepsilon B \\ 0, & a \not\varepsilon B \end{cases} \ , \tag{71}$$

the indicator function of B, then

$$T_tI_B(a) = P_a(t,B) \tag{72}$$

and satisfies the equation

$$\frac{\partial}{\partial t} P_a(t,B) = AP_a(t,B) \ , \quad P_a(0,B) = I_B(a) \ . \tag{73}$$

This defines the <u>fundamental solution</u> of (70) and corresponds to the case where the initial condition $x_0(\omega) = a$ is constant. It follows from (69), therefore, that the subset of solutions with nonrandom initial conditions completely characterizes the behavior of x_t. Henceforth, it will be assumed that $x_0(\omega) = a$ is a constant.

In the following development some specific types of Markov processes will be of interest. For example, $x_t(\omega)$ may be defined as the solution of the <u>Ito differential equation</u> [35]-[37]

$$dx_t = m(x_t)dt + \sigma(x_t)d\xi_t \tag{74}$$

where $\xi_t \in R^m$, $t \in R^+$, is a normalized Gaussian random process with independent increments. Equation 74 must be interpreted as an integral equation, but is usually written in the above differential form analogous to corresponding deterministic differential equations. It will be assumed that (74) is well-posed in the sense that it possesses unique solutions whose distribution functions are uniquely determined by some infinitesimal generator A.

Two particular cases are of interest. First, suppose

$\xi_t = w_t$, a normalized Wiener process. The infinitesimal generator then has the form

$$AV(a) \triangleq \mathcal{L}V(a) = m(a)'\nabla_a V(a) + \frac{1}{2}\,\mathrm{tr}[\sigma(a)'\,\nabla_{aa}V(a)\sigma(a)] \;. \tag{75}$$

This corresponds to the case where the stochastic disturbance is "white noise".

Another common form for the disturbance is "shot noise" which is obtained by letting $\xi_t = q_t$ be a normalized Poisson step process. The independent components q_i of q experience a jump in any interval of length Δt with probability $p_i\Delta t + o(\Delta t)$. The jump amplitude distribution is given by $P_i(dq_i)$ and is such that

$$\int_{q_i} q_i P_i(dq_i) = 0 \;, \qquad i=1,\ldots,m \;, \tag{76}$$

and

$$\int_{q_i} q_i^2 P_i(dq_i) = p_i^{-1} \;, \quad i=1,\ldots,m \;. \tag{77}$$

The corresponding infinitesimal generator is

$$AV(a) \triangleq \mathcal{D}V(a) = m(a)'\nabla_a V(a)$$

$$+ \sum_{i=1}^{m} \int_{q_i} [V(a+\sigma_{\cdot i}(a)q_i) - V(a)]\; p_i P_i(dq_i) \;, \tag{78}$$

where $\sigma_{\cdot i}$ denotes the i^{th} column of σ.

Note that if V has the quadratic form $V(a) = a'Pa$ then $\mathcal{L}V$ and $\mathcal{D}V$ will be identical. That is

$$\mathcal{L}a'Pa = \mathcal{D}a'Pa = 2m(a)'Pa + \mathrm{tr}[\sigma(a)'P\sigma(a)] \quad . \tag{79}$$

Therefore, results obtained for systems of the form (74) using quadratic V functionals will apply to systems with either Wiener or Poisson disturbances.

Another type of Markov process is one which is generated by the equation

$$\dot{x}_t = f(x_t, y_t) \tag{80}$$

where $y_t \in R^m$ is a jump Markov process taking values in the set $Y = \{y_i,\ i=1,\ldots,N\}$. The probability of a jump from y_i to y_j in any interval of length Δt is $p_{ij}\Delta t + o(\Delta t)$. In this case, in order to be considered as a Markov process, the system state x_t must be augmented by the disturbance y_t. The infinitesimal generator for this pair is given by

$$AV(a,y_i) \triangleq \mathcal{A}V(a,y_i) = f(a,y_i)'\nabla_a V(a,y_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} p_{ij}[V(a,y_j) - V(a,y_i)] \quad . \tag{81}$$

In this case, only part of the system state is of interest in a stability analysis, namely x_t. The definitions and theorems will take this possibility into account.

Given a Lyapunov functional V in the domain of A, the infinitesimal generator for some Markov process z_t, the stability theorems for stochastic systems are based on the equation

$$E_{z_0} V(z_t) = V(z_0) + E_{z_0} \int_0^t AV(z_t)dt \qquad (82)$$

which is called <u>Dynkin's formula</u> [35]-[37]. Then AV, which may be interpreted as the average rate of change of $V(z_t)$, performs a role analogous to that of $\dot{V}$ for deterministic systems.

There are several types of stochastic stability which could be considered. In the subsequent results the following definitions will be used. It is assumed that systems (74) and (80) are such that they possess the trivial solution $x_t(\omega) = 0$, $t \in R^+$.

<u>Definition 8</u>: The trivial solution is said to be <u>asymptotically stable in the large with probability one</u> (ASL w.p.1) if, given $x_0 = a$ (and $y_0 \in Y$),

(i) for every $\varepsilon > 0$ and $p > 0$ there is a $\delta > 0$ such that for any $a \in R^n$, $|a| < \delta$ implies

$$P\{\sup_{t \in R^+} |x_t| \geq \varepsilon \mid x_0 = a\} \leq p, \text{ and}$$

(ii) $|x_t| \to 0$ with probability one.∎

Definition 9: The trivial solution is said to be exponentially stable in the large with probability one (ESL w.p.1) if, given any $x_0 = a$ (and $y_0 \in Y$), condition (i) of Definition 8 is satisfied and

(ii)' for all $T \in R^+$ and $\varepsilon > 0$ there exist positive constants M and β such that

$$\mathcal{P}\{\sup_{t \geq T} |x_t| \geq \varepsilon \mid x_0 = a\} \leq Me^{-\beta T} .$$ ∎

These definitions pertain to the sample functions of x_t on R^+, which is probably of particular engineering interest since the observable behavior of a system is generally its sample functions. On the other hand moments might also be of interest and therefore the following definition is added.

Definition 10: The trivial solution is said to be exponentially stable in the large in the quadratic mean (ESL q.m.) if, given $x_0 = a$ (and $y_0 \in Y$),

(i) for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $a \in R^n$, $|a| < \delta$ implies

$$\sup_{t \in R^+} E_a |x_t|^2 \leq \varepsilon \quad , \text{ and}$$

(ii) there exist positive constants M and β such that

$E_a|x_t|^2 \leq Me^{-\beta t}$.∎

Stability results for large-scale stochastic systems will be based on the following theorems (see [36], [37]). The notation $V(a,\cdot)$ will be used to denote the possibility of the augmented state as discussed previously.

Theorem 8: Suppose there exist a Lyapunov functional V in the domain of A and three functions ψ_1, $\psi_2 \varepsilon KR$, $\psi_3 \varepsilon K$ such that

(i) $\psi_1(|a|) \leq V(a,\cdot) \leq \psi_2(|a|)$, and

(ii) $AV(a,\cdot) \leq -\psi_3(|a|)$,

for all $a \varepsilon R^n$ (and $y_0 \varepsilon Y$). Then the trivial solution of (74) (or (80)) is ASL w.p.1.∎

Theorem 9: If in Theorem 8, $\psi_1(r) = c_1r^2$, $\psi_2(r) = c_2r^2$, and $\psi_3(r) = c_3r^2$ where c_1, c_2 and c_3 are positive constants, then the trivial solution of (74) (or (80)) is ESL w.p.1 and ESL q.m.∎

For other results see [36]-[41].

VI. MAIN RESULTS: LARGE-SCALE STOCHASTIC SYSTEMS

The large-scale versions of stochastic systems (74) and (80) will now be considered in a fashion similar to that of Chapter IV.

The isolated subsystems for systems of the type (74) are

$$\mathcal{S}_i \; : \; dz_t^i = m_i(z_t^i)dt + \sigma_{ii}(z_t^i)d\xi_t^i \; , \quad i=1,\ldots,\ell \; , \qquad (83)$$

where $z_t^i \in R^{n_i}$, and $\xi_t^i \in R^{m_i}$ is a normalized Gaussian random process with independent increments for each $i=1,\ldots,\ell$. Assume subsystems (83) possess trivial solutions.

The isolated subsystems for systems of the type (80) are

$$\mathcal{S}_i \; : \; \dot{z}_t^i = f_i(z_t^i, y_t^i) \; , \quad i=1,\ldots,\ell \; , \qquad (84)$$

where $z_t^i \in R^{n_i}$, and $y_t^i \in R^{m_i}$ is a jump Markov process taking values in the set $Y_i = \{y_j^i, \; j=1,\ldots,N_i\}$ for each $i=1,\ldots,\ell$. Assume subsystems (84) possess trivial solutions.

The subsystems $\mathcal{S}_i$ may be analyzed by Lyapunov's direct method as indicated in the previous chapter. As with the subsystems of Chapter IV the resulting information is summarized in the following properties.

<u>Definition 11</u>: An isolated subsystem $\mathcal{S}_i$ is said to possess <u>Property C</u> if there exist a Lyapunov function V_i in the domain of A_i (the infinitesimal generator of $\mathcal{S}_i$), three functions ψ_{i1}, $\psi_{i2} \in KR$, $\psi_{i3} \in K$, and a real constant σ_i such that

(i) $\psi_{i1}(|a_i|) \leq V_i(a_i,\cdot) \leq \psi_{i2}(|a_i|)$, and

(ii) $A_i V_i(a_i,\cdot) \leq \sigma_i \psi_{i3}(|a_i|)$,

for all $a_i \in R^{n_i}$ (and $y_0^i \in Y_i$).∎

<u>Definition 12</u>: If in Definition 11, $\psi_{i1}(r) = c_{i1}r^2$, $\psi_{i2}(r) = c_{i2}r^2$, $\psi_{i3}(r) = r^2$, where c_{i1} and c_{i2} are positive constants, then isolated subsystem $\mathcal{S}_i$ is said to possess <u>Property D</u>.∎

If $\sigma_i < 0$, then Definitions 11 and 12 correspond to the hypotheses of stability Theorems 8 and 9. Therefore σ_i plays the same role here as it did in Chapter IV, being a measure of the degree of stability of $\mathcal{S}_i$.

The subsystems $\mathcal{S}_i$ of the form (83) will be interconnected to form a composite system as follows

$$\mathcal{S} : dz_t^i = m_i(z_t^i)dt + g_i(x_t)dt + \sum_{j=1}^{\ell} \sigma_{ij}(z_t^j)d\xi_t^j \ , \qquad (85)$$

$i=1,\ldots,\ell$, where $x_t = (z_t^i)$, $i=1,\ldots,\ell$. The disturbances ξ_t^i are assumed to be independent. Letting $m(x) =$

$(m_i(z^i) + g_i(x))$, $i=1,\ldots,\ell$, $\sigma(x) = ((\sigma_{ij}(z^j)))$, $i,j=1,\ldots,\ell$, and $\xi_t = (\xi_t^j)$, $j=1,\ldots,\ell$, then (85) can be expressed equivalently as

$$\mathcal{S}: dx_t = m(x_t)dt + \sigma(x_t)d\xi_t \tag{86}$$

which is identical to (74). It is assumed that (86) possesses the trivial solution $x_t = 0$, $t \varepsilon R^+$, and that composite system $\mathcal{S}$ and its isolated subsystems $\mathcal{S}_i$ are well-posed (see [35]-[37]).

Subsystem $\mathcal{S}_i$ of the form (84) will be interconnected to form a composite system as follows

$$\mathcal{S}: \dot{z}_t^i = f_i(z_t^i, y_t^i) + g_i(x_t, y_t) \tag{87}$$

$i=1,\ldots,\ell$, where $x_t = (z_t^i)$, $i=1,\ldots,\ell$. The disturbances y_t^i are assumed independent. Letting $f(x,y) = (f_i(z^i,y^i) + g_i(x,y))$, $i=1,\ldots,\ell$, where $y = (y^i)$, $i=1,\ldots,\ell$, then (87) can be expressed equivalently as

$$\mathcal{S}: \dot{x}_t = f(x_t, y_t) \tag{88}$$

which is identical to (80). It is assumed that (88) possesses the trivial solution $x_t = 0$, $t \varepsilon R^+$, and that composite system $\mathcal{S}$ and its isolated subsystems $\mathcal{S}_i$ are well-posed (see [37], [38]).

Note that composite systems (85) and (87) contain stochastic disturbances not only in the subsystems but in the interconnecting structure as well. The following results will make it possible to determine the influence of these disturbances on the stability of the composite system.

The proofs of the following theorems may be found in Appendix B.

Theorem 10: Assume that composite system $\mathcal{S}$ (described by (85) or (87)) satisfies the following conditions:

(i) in (85), $\xi_t = (\xi_t^j)$, $j=1,\ldots,\ell$, is either a Wiener process or a Poisson process, and there are no stochastic disturbances in the interconnecting structure, i.e., $\sigma_{ij}(z^j) \equiv 0$, $i \neq j$;

(ii) each isolated subsystem $\mathcal{S}_i$ possesses Property C;

(iii) given the Lyapunov functionals V_i and comparison functions ψ_{i3}, $i=1,\ldots,\ell$, of hypothesis (ii), there exist real constants b_{ij}, $i,j=1,\ldots,\ell$, such that

$$g_i(a,\cdot)' \nabla_{a_i} V_i(a_i,\cdot)$$

$$\leq [\psi_{i3}(|a_i|)]^{1/2} \sum_{j=1}^{\ell} b_{ij} [\psi_{i3}(|a_j|)]^{1/2} \tag{89}$$

for all $a \in R^n$ (and $y \in Y$); and

(iv) there exist positive constants α_i, $i=1,\ldots,\ell$, such that the test matrix $S = ((s_{ij}))$, $i,j=1,\ldots,\ell$, defined by

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i+b_{ii}) & , \quad i=j \\ \frac{1}{2}(\alpha_i b_{ij}+\alpha_j b_{ji}) & , \quad i\neq j \end{cases} \tag{90}$$

is negative definite.

Then the trivial solution of $\mathcal{S}$ is ASL w.p.1.∎

Note that this result is quite similar to Theorem 3 of Chapter IV. In fact, the test matrices are identical and therefore all observations following Theorems 3 and 4 in Chapter IV are applicable. This includes the results involving M-matrices. Theorems 5-7 are readily adaptable in an obvious manner to apply to the above large-scale stochastic systems.

The case where composite system $\mathcal{S}$ described by (85) has nonzero interconnection disturbances remains to be considered. This will introduce additional terms into the test matrix.

<u>Theorem 11</u>: Assume that composite system $\mathcal{S}$ (described by (85)) satisfies the following conditions:

(i) in (85), $\xi_t = (\xi_t^i)$, $i=1,\ldots,\ell$, is a Wiener process and in general the interconnecting structure disturbances are nonzero;

(ii) each isolated subsystem $\mathcal{S}_i$ possesses Property C;

(iii) given the Lyapunov functionals V_i and comparison functions ψ_{i3}, $i=1,\ldots,\ell$, of hypothesis (ii), there exist real constants b_{ij}, $i,j=1,\ldots,\ell$, such that

$$g_i(a)'\nabla_{a_i}V_i(a_i)$$

$$\leq [\psi_{i3}(|a_i|)]^{1/2}\sum_{j=1}^{\ell} b_{ij}[\psi_{j3}(|a_j|)]^{1/2} \qquad (91)$$

for all $a \in R^n$;

(iv) for each V_i, $i=1,\ldots,\ell$, there is a positive constant e_i such that

$$a_i'\nabla_{a_i a_i}V_i(a_i)a_i \leq e_i\ |a_i|^2 \qquad (92)$$

for all $a_i \in R^{n_i}$, $i=1,\ldots,\ell$;

(v) for each σ_{ij}, $i,j=1,\ldots,\ell$, $i\neq j$, there exists a constant $d_{ij} \geq 0$ such that

$$\|\sigma_{ij}(a_j)\|_m^2 \leq d_{ij}\psi_{j3}(|a_j|) \qquad (93)$$

for all $a_j \in R^{n_j}$, $j=1,\ldots,\ell$; and

(vi) there exist positive constants α_i, $i=1,\ldots,\ell$, such that the test matrix $S = (\!(s_{ij})\!)$, $i,j=1,\ldots,\ell$

defined by

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + b_{ii}) + \frac{1}{2} \sum_{\substack{k=1 \\ k \neq i}}^{\ell} \alpha_k e_k d_{ki} , & i=j \\ \frac{1}{2}[\alpha_i b_{ij} + \alpha_j b_{ji}] , & i \neq j \end{cases} \tag{94}$$

is negative definite.

Then the trivial solution of $\mathcal{S}$ is ASL w.p.l.∎

Theorem 12: If in Theorems 10 and 11 each isolated subsystem $\mathcal{S}_i$ possesses Property D, then the trivial solution of $\mathcal{S}$ is ESL w.p.l and ESL q.m.∎

Finally, by taking advantage of the equivalence between the infinitesimal generators $\mathcal{L}$ and $\mathcal{D}$ for quadratic Lyapunov functionals, as expressed in (79), the above results can be extended as follows.

Theorem 13: Assume that composite system $\mathcal{S}$ (described by (85)) satisfies the following conditions:

(i) in (85), $\xi_t = (\xi_t^i)$, $i=1,\dots,\ell$, is either a Wiener process or a Poisson process, and in general the interconnecting structure disturbances are nonzero,

(ii) each isolated subsystem $\mathcal{S}_i$ possesses Property D with $V_i(a_i) = a_i' P_i a_i$, where P_i is a positive definite $n_i \times n_i$ matrix;

(iii) given the matrices P_i of hypothesis (ii), there exist real constants b_{ij}, $i,j=1,\ldots,\ell$, such that

$$g_i(a)'P_i a_i \leq \frac{1}{2} |a_i| \sum_{j=1}^{\ell} b_{ij} |a_j| \tag{95}$$

for all $a \varepsilon R^n$;

((iv) for each σ_{ij}, $i,j=1,\ldots,\ell$, $i \neq j$, there exists a constant $d_{ij} \geq 0$ such that

$$\|\sigma_{ij}(a_j)\|_m^2 \leq d_{ij} |a_j|^2 \tag{96}$$

for all $a_j \varepsilon R^{n_j}$, $j=1,\ldots,\ell$; and

(v) there exist positive constants α_i, $i=1,\ldots,\ell$, such that the test matrix $S = ((s_{ij}))$, $i,j=1,\ldots,\ell$, defined by

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + b_{ii}) + \sum_{\substack{k=1 \\ k \neq i}}^{\ell} \alpha_k \lambda_M(P_k) d_{ki}, & i=j \\ \frac{1}{2}(\alpha_i b_{ij} + \alpha_j b_{ji}), & i \neq j \end{cases} \tag{97}$$

is negative definite.

Then the trivial solution of $\mathcal{S}$ is ESL w.p.1 and ESL q.m.∎

Note that in Theorems 11-13 the interconnection disturbance terms d_{ki}, $k,i=1,\ldots,\ell$, $k \neq i$, which express the magnitude of the disturbances, occur on the diagonal of the test matrix. Their effect on the negative definiteness of S is to make more restrictive the conditions on the remaining parameters of the matrix. That is, the disturbances have a degrading influence on the stability of the composite system $\mathcal{S}$.

As in Chapter IV, these results are also of a weak coupling nature. In addition, the stabilization procedure suggested there is also applicable here.

Attempts at applying the theory of M-matrices to Theorems 11-13 are not fruitful due to the addition diagonal terms.

The following examples will illustrate the results obtained in this section. The first example is a stochastic version of the indirect control problem. The second example is of a nonlinear system with random parameters. The last example is included as a demonstration of the non-redundancy of various results.

<u>Example 5</u>: Consider the following version of the indirect control problem with shot noise

$$\left.\begin{aligned} dz_t^1 &= Az_t^1 dt + bf(z_t^2)dt + \sum_{j=1}^{2} \sigma_{1j}(z_t^j)dq_t^j \\ dz_t^2 &= [-\rho z_t^2 - rf(z_t^2)]dt + c' z_t^1 + \sum_{j=1}^{2} \sigma_{2j}(z_t^j)dq_t^j \end{aligned}\right\} \quad (98)$$

where $z_t^1 \varepsilon R^{n_1}$ and $z_t^2 \varepsilon R$, $t \varepsilon R^+$. The matrix A is assumed to be stable, b and c are n_1-vectors, the scalars ρ and r are positive, the matrix valued functions σ_{ij} are assumed to satisfy the inequality (96), and q_t^i, i=1,2, are independent Poisson processes. The nonlinear function f is continuous and satisfies the condition

$$0 \leq a_2 f(a_2) \leq K a_2^2 \tag{99}$$

for some constant $K > 0$.

The isolated subsystems are chosen as

$$\left.\begin{aligned} \mathcal{S}_1 &: dz_t^1 = Az_t^1 dt + \sigma_{11}(z_t^1)dq_t^1 \\ \mathcal{S}_2 &: dz_t^2 = [-\rho z_t^2 - rf(z_t^2)]dt + \sigma_{22}(z_t^2)dq_t^2 \end{aligned}\right\} . \tag{100}$$

Since A is a stable matrix, there exists a positive definite symmetric matrix P such that $A'P + PA \triangleq -Q$ is negative definite. Choosing $V_1(a_1) = a_1'Pa_1$, one has for $\mathcal{S}_1$, $e_1 = 2\lambda_M(P)$,

$$\lambda_m(P)|a_1|^2 \leq V_1(a_1) \leq \lambda_M(P)|a_1|^2 \tag{101}$$

and

$$\begin{aligned} \mathcal{D} V_1(a_1) &= a_1'(A'P+PA)a_1 + \mathrm{tr}[\sigma_{11}(a_1)P\sigma_{11}(a)] \\ &\leq -a_1'Qa_1 + \lambda_M(P)\,\|\sigma_{11}(a_1)\|_m^2 \\ &\leq [-\lambda_m(Q) + \lambda_M(P)d_{11}]|a_1|^2 \quad . \end{aligned} \tag{102}$$

For $\mathcal{S}_2$ choose $V_2(a_2) = a_2^2$ so that $e_2 = 2$ and

$$\mathcal{D}V_2(a_2) = -2\rho a_2^2 - 2ra_2 f(a_2) + \|\sigma_{22}(a_2)\|_m^2$$

$$\leq (-2\rho + d_{22})|a_2|^2 \quad . \tag{103}$$

Isolated subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ both possess Property D with

$$\sigma_1 = -\lambda_m(Q) + \lambda_M(P)d_{11} \quad , \tag{104}$$

and

$$\sigma_2 = -2\rho + d_{22} \ . \tag{105}$$

For the interconnections one has

$$2g_1(a)'Pa_1 = 2f(a_2)b'Pa_1 \leq 2K|b|\lambda_M(P)|a_1||a_2| \tag{106}$$

and

$$2g_2(a)'a_2 = 2a_1'ca_2 \leq 2|c||a_1||a_2| \quad , \tag{107}$$

giving

$$b_{11} = b_{22} = 0 \ , \ b_{12} = 2K|b|\lambda_M(P) \ , \ b_{21} = 2|c| \ . \tag{108}$$

By choosing $\alpha_1 = 1/\lambda_M(P)$ and $\alpha_2 = 1$ matrix S of Theorem 13 becomes

$$S = \begin{bmatrix} -\frac{\lambda_m(Q)}{\lambda_M(P)} + d_{11} + d_{21} & |c| + K|b| \\ K|b| + |c| & -2\rho + d_{22} + d_{12} \end{bmatrix} \tag{109}$$

which is negative definite if and only if

$$\rho > \frac{1}{2}(d_{22} + d_{12}) \tag{110}$$

and

$$K < |b|^{-1}\{[\frac{\lambda_m(Q)}{\lambda_M(P)} - d_{11} - d_{22}]^{1/2}[2\rho - d_{22} - d_{12}]^{1/2} - |a|\} . \tag{111}$$

That is, if (110) and (111) hold then composite system (98) is ESL w.p.1 and ESL i.p. The weak coupling nature of this result and the degrading effect of the noise are clearly evident in these formulas.∎

<u>Example 6</u>: Consider the system

$$\dot{x}_t = [A(x_t) + N(t)]x_t \quad , \quad x_0 = a \quad , \tag{112}$$

where $x_t \in R^{\ell}$, $t \in T$, $A(x)$ is an $\ell \times \ell$ array of continuous bounded scalar functions $a_{ij}(x)$, $i,j=1,\ldots,\ell$, and $N(t)$ is a random matrix of independent wide-band, zero mean, Gaussian random processes $\bar{\sigma}_{ij}\, n_{ij}(t)$, $i,j=1,\ldots,\ell$. System (112) may be considered as a nonlinear system with random parameters.

By letting N(t) approach a white noise matrix, Equation 112 may be replaced by an equivalent Ito differential equation as follows. Let $\sigma(x)$ be the $\ell \times \ell^2$ array of $1 \times \ell$ submatrices $\sigma_{ij}(x^j)$ where

$$\sigma_{ij}(x^j)' = (\delta_{ij}\bar{\sigma}_{ij}x^j) \quad , \quad j=1,\ldots,\ell, \tag{113}$$

and let $\nu(t)$ be the $\ell^2 \times 1$ vector given by

$$\nu(t)' = (n_{11}(t), \ldots, n_{\ell 1}(t), n_{12}(t), \ldots, n_{\ell\ell}(t)) \quad . \tag{114}$$

Then (112) may be written equivalently as

$$\dot{x}_t = A(x_t)x_t + \sigma(x_t)\nu(t) \quad . \tag{115}$$

Following rules of transformation (see e.g. [35]), (115) is replaced by the Ito differential equation

$$dx_t = m(x_t)dt + \sigma(x_t)dw_t \tag{116}$$

where $\sigma(x)$ is as above, $w_t \in R^{\ell^2}$ is a normalized Wiener process, and $m(x) = (m_i(x))$, $i=1,\ldots,\ell$, where

$$m_i(x) = \sum_{j=1}^{\ell} a_{ij}(x)x^j + \frac{1}{2}\bar{\sigma}_{ii}^2 x^i \quad , \quad i=1,\ldots,\ell \quad . \tag{117}$$

The extra term arises from the second order properties of Ito calculus.

From (113) one obtains the following

$$\|\sigma_{ij}(x^j)\|_m^2 = \bar{\sigma}_{ij}^2 \; |x^j|^2 \quad , \quad i,j=1,\ldots,\ell \; . \tag{118}$$

Therefore, one has $d_{ij} = \bar{\sigma}_{ij}^2$.

Choosing the subsystems as

$$\mathcal{S}_i \; : \; dx_t^i = \frac{1}{2}\,\bar{\sigma}_{ii}^2 \; x_t^i dt + \sigma_{ii}(x_t^i)dw_t^i \tag{119}$$

where $w_t^i \, \varepsilon R^\ell$, and letting $V_i(x^i) = \frac{1}{2}\,|x^i|^2$, then $e_i = 1$ and

$$\mathcal{L}_i V_i(x^i) = \frac{1}{2}\,\bar{\sigma}_{ii}^2 \; |x_i|^2 + \frac{1}{2}\; \mathrm{tr}[\sigma_{ii}(x^i)'\,\sigma_{ii}(x^i)]$$

$$\leq \bar{\sigma}_{ii}^2 \; |x^i|^2 \quad , \quad i=1,\ldots,\ell \; . \tag{120}$$

Therefore, the subsystems possess Property D with

$$\sigma_i = \bar{\sigma}_{ii}^2 \quad , \quad i=1,\ldots,\ell \; . \tag{121}$$

In addition, since

$$g_i(x)'\nabla_{x^i} V_i(x^i) = \sum_{j=1}^{\ell} a_{ij}(x)x^j x^i$$

$$\leq \sup_{R^\ell} a_{ii}(x)|x^i|^2 + |x^i| \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sup_{R^\ell} |a_{ij}(x)|\,|x^j| \tag{122}$$

then

$$b_{ii} = \sup_{R^\ell} a_{ii}(x) \; , \; b_{ij} = \sup_{R^\ell} |a_{ij}(x)| \; , \quad i\neq j \quad , \quad i,j=1,\ldots,\ell \; . \tag{123}$$

The test matrix $S = ((s_{ij}))$, $i,j=1,\ldots,\ell$ is given by

$$s_{ij} = \begin{cases} \alpha_i(\sup_{R^\ell} a_{ii}(x) + \bar{\sigma}_{ii}^2) + \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{\ell} \alpha_k \bar{\sigma}_{ki}^2 \ , & i=j \\ \frac{1}{2}(\alpha_i \sup_{R^\ell} |a_{ij}(x)| + \alpha_j \sup |a_{ji}(x)|) \ , & i\neq j \end{cases} \quad (124)$$

and if S is negative definite, system (112) will be ESL w.p.1 and ESL q.m. by Theorem 12.

If in particular $\bar{\sigma}_{ij} = 0$, $i\neq j$, $i,j=1,\ldots,\ell$ (that is, only the diagonal elements of A(x) are disturbed), then Theorem 10 may be applied. In fact, since

$$\sup_{R^\ell} |a_{ij}(x)| \geq 0$$

the M-matrix results are applicable as well. In analogy to Theorem 6 one obtains the condition

$$\sup_{R^\ell} a_{ii}(x) < -\bar{\sigma}_{ii}^2 - \sum_{\substack{j=1\\j\neq i}}^{\ell} \left(\frac{\lambda_j}{\lambda_i}\right) \sup_{R^\ell} |a_{ij}(x)| \leq 0 \ , \quad (125)$$

$i=1,\ldots,\ell$, for some constants $\lambda_i > 0$, $i=1,\ldots,\ell$. Again, note the weak coupling and noise degradation implied by this result.∎

<u>Example 7</u>: The following simple example will address a couple of points which have arisen from the previous

discussion. These will be indicated as they occur. The system to be considered is

$$\left.\begin{aligned} \dot{z}_t^1 &= - y_t^1 z_t^1 - (z_t^1)^3 - z_t^1 |z_t^2| \\ \dot{z}_t^2 &= - y_t^2 z_t^2 |z_t^2| + (z_t^1)^2 z_t^2 \end{aligned}\right\} \tag{126}$$

where $z_t^i \varepsilon R$, $i=1,2$, $y_t^1 \varepsilon Y_1 = \{1,-1\}$ and $y_t^2 \varepsilon Y_2 = \{1,3/2,2\}$. The jump probability coefficients are $p_{12}^1 = 1$, $p_{21}^1 = 5$, and $p_{jk}^2 = 1/2$, $j \neq k$, $j,k=1,2,3$.

Choose the isolated subsystems as

$$\left.\begin{aligned} \mathcal{S}_1 &: \dot{z}_t^1 = - y_t^1 z_t^1 - (z_t^1)^3 \\ \mathcal{S}_2 &: \dot{z}_t^2 = - y_t^2 z_t^2 |z_t^2| \end{aligned}\right\} \quad , \tag{127}$$

and let

$$V_1(a_1,y^1) = \begin{cases} a_1^2 , & y^1 = 1 \\ 2a_1^2 , & y^1 = -1 \end{cases} \quad , \tag{128}$$

$$V_2(a_2,y^2) = |a_2| \quad . \tag{129}$$

Then

$$a_1^2 \leq V_1(a_1,y^1) \leq 2a_1^2 \quad , \tag{130}$$

$$\mathcal{G}_1 V_1(a_1,1) = 2a_1(-\ a_1 - a_1^3) + [2a_1^2 - a_1^2]$$

$$= -\ a_1^2 - 2a_1^4 \leq -\ 2a_1^4 \quad , \tag{131}$$

$$\mathcal{G}_1 V_1(a_1,-1) = 4a_1(a_1 - a_1^3) + 5[a_1^2 - 2a_1^2]$$

$$= -\ a_1^2 - 4a_1^4 \leq -\ 4a_1^4 \tag{132}$$

and

$$\mathcal{G}_2 V_2(a_2,y_i^2) = \text{sgn}(a_2)(-\ y_i^2 a_2 |a_2|)$$

$$= -\ y_i^2 a_2^2 \leq -\ a_2^2 \ , \quad i=1,2,3 \quad . \tag{133}$$

Letting $\psi_{11}(r) = r^2$, $\psi_{12}(r) = 2r^2$, $\psi_{13}(r) = r^4$, $\psi_{21}(r) = \psi_{22}(r) = |r|$ and $\psi_{23}(r) = r^2$, it follows that subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ possess Property C with

$$\sigma_1 = -2 \quad , \quad \sigma_2 = -1 \quad , \tag{134}$$

and that they do not possess Property D. The implication of this is that asymptotic stability may be shown, but not exponential stability. This alleviates the suspicion that only exponentially stable systems can be treated by the present method, as it may have appeared from previous examples.

For the interconnections one obtains

$$g_1(a)'\nabla_{a_1} V_1(a_1,1) = 2a_1(-a_1|a_2|) = -2|a_1|^2|a_2| \quad , \quad (135)$$

$$g_1(a)'\nabla_{a_1} V_1(a_1,-1) = 4a_1(-a_1|a_2|) = -4|a_1|^2|a_2| \quad , \quad (136)$$

and

$$g_2(a)'\nabla_{a_2} V_2(a_2,y_i^2) = \text{sgn}(a_2)(a_1^2a_2) = |a_2||a_1|^2 \ . \quad (137)$$

Therefore, since $[\psi_{13}(|a_1|)]^{1/2} = |a_1|^2$ and $[\psi_{23}(|a_2|)]^{1/2} = |a_2|$, one has

$$b_{11} = b_{22} = 0 \ , \quad b_{12} = -2 \quad \text{and} \quad b_{21} = 1 \ . \quad (138)$$

Note that $b_{12} < 0$ so that the theory of M-matrices is not applicable here in the previous manner. This shows that the theorems with negative definite test conditions are more general than the theorems using M-matrix conditions. This also was not evident in previous examples.

Using the constants obtained above and choosing $\alpha_1 = \alpha_2 = 1$, the test matrix of Theorem 10 is

$$S = \begin{bmatrix} -2 & -1/2 \\ & \\ -1/2 & -1 \end{bmatrix} \quad , \quad (139)$$

which is negative definite. Therefore interconnected system (126) is ASL w.p.1.

An interesting aspect of this example is that $\mathcal{S}_1$

possesses a structure for $y_t^1 = -1$ which in the deterministic case would be unstable. The importance of allowing V_i to depend on the augmented state is illustrated in this case, since without this possibility V_i could not be chosen to make $\mathcal{G}_i V_i$ negative definite.∎

Note finally that the deterministic versions of Examples 5-7 could be analyzed using the results of Chapter IV. In particular, Example 7 shows that negative interconnection constants b_{ij}, $i \neq j$, are possible in Theorems 3 and 4, and that it is possible to have systems to which Theorem 3 applies, but not Theorem 4.

VII. CONCLUDING OBSERVATIONS

A number of remarks concerning the results presented in this dissertation and their relationship to other results on large-scale systems are in order. The present results represent a collection of selected results from papers [42] through [47] which were chosen to demonstrate from an overall view one method of handling the Lyapunov stability analysis of large-scale systems. This is essentially the technique of applying comparison functions and weighted-sum Lyapunov functionals to interconnected systems in a fashion which allows one to express weak coupling stability results in terms of conditions on a test matrix. This had been accomplished for systems described by ordinary differential equations, and a few other special cases in such references as [7]-[11].

The present method, which is essentially a scalar Lyapunov method, is to be contrasted with a different approach involving vector Lyapunov functions as introduced by Bellman [48]. The latter method, although it also employs the concept of an interconnected system, departs from the scalar method in the manner in which the information regarding the subsystems and interconnecting structure is used. With the vector Lyapunov function method a lower order comparison system is obtained, the stability of which is used to imply, via a comparison principle, the stability of the original composite system. Any test matrices that arise in the analysis (if any) simply define the dynamics of the comparison system. Since the present approach and the vector approach are different and yield different results, they should be viewed as being complementary results. Among the references available for vector Lyapunov function results are [1]-[6] and [49], [50], the latter two being contributed to by this author wherein a new comparison principle for stochastic systems is obtained.

The theory presented in this dissertation represents an extension of many previous results on large-scale systems. Specifically, the principle contributions made by this author have been the application of the method to general dynamical systems on Banach spaces, the extension of the method to a variety of stochastic systems and to stochastic systems with disturbances in the interconnecting structure, the introduction of a degree of stability (or instability) parameter into the subsystem characterization, the demonstration of additional M-matrix test procedures, the statement of more general exponential stability results, and the development of a margin of stability parameter for large-scale systems. Due to the more general approach presented here many of the results in [7]-[11] can be shown to be special cases of the present results.

It should be noted that the present results have been restricted to autonomous systems. This was merely to simplify the demonstration. In fact, the extension to nonautonomous systems is generally not difficult. These nonautonomous cases are considered in [42]-[47]. The cited papers also contain other results which have not been included here, for example: corollaries exploiting specific interconnecting structure forms, theorems involving a different type of bound on the interconnecting structure, and further examples to illustrate the present method.

A number of topics for further research present themselves when the present results are studied. Perhaps the most important of these is to find a technique for the stability analysis of large-scale systems which does not yield weak coupling results. This is the main source of conservatism in present results, and it is responsible for the general observation that, the finer one decomposes a large-scale system, the more conservative the results tend to become. This is a disadvantage which forces a compromise between the present method and conventional Lyapunov methods.

A closely related problem is that of dealing with unstable subsystems. As yet no results have been obtained for this case without the necessity of providing local stabilizing feedback around unstable subsystems through the interconnecting structure.

In theorems requiring the choice of arbitrary constants (e.g., the α_i's of Theorem 3) it is of interest whether or not some computational technique exists for choosing these constants. It seems likely that an iterative procedure could be implemented in a computer program which would find such constants quickly. This would be especially important if the number of constants to be chosen was large. One might also have some criterion by which the choice of optimum constants could be made. This choice might, for example, be the one which minimizes the conservativeness of the stability condition.

Finally, it would be of great interest to find actual specific physical problems for which the present theory would provide a useful solution. This would do much towards making these results attractive tools for analysis. Example 4, the coupled core nuclear reactor system, was an attempt at providing such a problem. Each of the constants and functions in that example may supposedly be determined by experimental or theoretical means. This would be a case therefore where the stability results obtained would be physically verifiable.

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IX. ACKNOWLEDGMENTS

I am grateful to many people for their assistance and encouragement in this endeavor. Special thanks goes to Dr. Anthony N. Michel who provided the necessary inspiration and motivation for the project, and who supported me through its completion.

This work is dedicated to Linda, my wife, in deepest appreciation for her invaluable companionship during this period.

X. APPENDIX A

The proofs of the main results of Chapter IV regarding the stability of large-scale systems on Banach spaces are as follows.

Proof of Theorem 3: By hypothesis (i) of Theorem 3 and the definition of Property A, there exist admissible Lyapunov functionals V_i for each isolated subsystem $\mathcal{S}_i$. By Definition 4, the sets $\overline{Q}_{m_i}$ are closed and bounded. Choose as a Lyapunov functional for the composite system $\mathcal{S}$ the weighted sum

$$V(a) = \sum_{i=1}^{\ell} \alpha_i V_i(a_i) \tag{140}$$

where the constants $\alpha_i > 0$, $i=1,\ldots,\ell$, are as yet undetermined. If $m \triangleq \min_i(\alpha_i m_i)$, then $a \in Q_m$ implies

$$V_i(a_i) \leq V(a)/\alpha_i < m/\alpha_i \leq m_i \tag{141}$$

or

$$a_i \in Q_{m_i} \;, \quad i=1,\ldots,\ell \quad . \tag{142}$$

That is, $\overline{Q}_m$ is bounded. In addition, if Theorems 1 or 2 are applied to $\mathcal{S}$ with m as specified, then a_i, $i=1,\ldots,\ell$, are guaranteed to be such that hypotheses (i) and (ii) are applicable.

The continuity of V in (140) follows from the continuity of V_i, $i=1,\ldots,\ell$. Furthermore, applying Definition 4 to (140) yields

$$V(a+x) - V(a) = \sum_{i=1}^{\ell} \alpha_i [V_i(a_i+z^i) - V(a_i)]$$

$$\leq \sum_{i=1}^{\ell} \alpha_i [\nabla V_i(a_i,z^i) + o(\|z^i\|_i)]$$

$$= \sum_{i=1}^{\ell} \alpha_i \nabla V_i(a_i,z^i) + o(\|x\|) \quad . \qquad (143)$$

That is,

$$\nabla V(a,x) = \sum_{i=1}^{\ell} \alpha_i \nabla V_i(a_i,z^i) \quad . \qquad (144)$$

Since ∇V_i, $i=1,\ldots,\ell$, are linear and continuous in z^i, uniformly with respect to $a_i \varepsilon Q_{m_i}$, it follows that ∇V is linear and continuous in x, uniformly with respect to $a \varepsilon Q_m$.

Finally since $V_i(0) = 0$, $i=1,\ldots,\ell$, then $V(0) = 0$ and therefore V is admissible by Definition 4.

By Property A there exist functions ψ_{i1}, $\psi_{i2} \varepsilon KR$ for each subsystem $\mathcal{S}_i$ such that for $a_i \varepsilon \overline{Q}_{m_i}$, $i=1,\ldots,\ell$,

$$\sum_{i=1}^{\ell} \alpha_i \psi_{i1}(\|a_i\|_i) \leq V(a) \leq \sum_{i=1}^{\ell} \alpha_i \psi_{i2}(\|a_i\|_i) \quad . \qquad (145)$$

The summations on the left and right are both positive definite, decrescent and radially unbounded. Therefore by the properties of such functions (see e.g. [12]) each can be bounded above and below by functions of class KR. That is, there exist ψ_1, $\psi_2 \in KR$ such that

$$\psi_1(\|a\|) \leq \sum_{i=1}^{\ell} \alpha_i \psi_{i1}(\|a_i\|_i) \quad , \tag{146}$$

and

$$\sum_{i=1}^{\ell} \alpha_i \psi_{i2}(\|a_i\|_i) \leq \psi_2(\|a\|) \quad , \tag{147}$$

and therefore

$$\psi_1(\|a\|) \leq V(a) \leq \psi_2(\|a\|) \tag{148}$$

for all $a \in \overline{Q}_m$.

Finally, by the linearity of ∇V and by hypotheses (i) and (ii) of Theorem 3, one has for $a \in \overline{Q}_m \cap D(A)$

$$\begin{aligned}\nabla V(a,Aa) &= \sum_{i=1}^{\ell} \alpha_i \nabla V_i(a_i, A_i a) \\ &= \sum_{i=1}^{\ell} \alpha_i \nabla V_i(a_i, F_i a_i + G_i a) \\ &= \sum_{i=1}^{\ell} \alpha_i [\nabla V_i(a_i, F_i a_i) + \nabla V_i(a_i, G_i a)]\end{aligned}$$

$$\leq \sum_{i=1}^{\ell} \alpha_i \{\sigma_i \psi_{i3}(\|a_i\|_i)$$

$$+ [\psi_{i3}(\|a_i\|_i)]^{1/2} \sum_{j=1}^{\ell} b_{ij} [\psi_{j3}(\|a_j\|_j)]^{1/2}\}$$

$$= \sum_{i=1}^{\ell} \alpha_i (\sigma_i + b_{ii}) \{[\psi_{i3}(\|a_i\|_i)]^{1/2}\}^2$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \tfrac{1}{2}(\alpha_i b_{ij} + \alpha_j b_{ji}) [\psi_{i3}(\|a_i\|_i)]^{1/2} [\psi_{j3}(\|a_j\|_j)]^{1/2}$$

$$= u' \, S \, u \qquad (149)$$

where $u = ([\psi_{i3}(\|a_i\|_i)]^{1/2})$, $i=1,\ldots,\ell$, and S is the test matrix of hypothesis (iii). Note that the symmetry of the matrix S is deliberate.

By hypothesis (iii) the positive constants α_i, $i=1,\ldots,\ell$, may now be chosen such that S is negative definite. Therefore $\lambda_M(S)$ is a negative real number so that

$$u'Su \leq \lambda_M(S) \; |u|^2 = \lambda_M(S) \sum_{i=1}^{\ell} \psi_{i3}(\|a_i\|_i) \; . \qquad (150)$$

Since the term on the right is negative definite, it may be bounded by the negative of some function of class K. That is, there exists $\psi_3 \varepsilon K$ such that

$$\lambda_M(S) \sum_{i=1}^{\ell} \psi_{i3}(\|a_i\|_i) \leq -\psi_3(\|a\|) \quad . \tag{151}$$

It follows from (149)-(151) that

$$\nabla V(a,Aa) \leq - \psi_3(\|a\|) \quad . \tag{152}$$

All conditions of Theorem 1 have been satisfied for $\mathcal{S}$ and therefore the conclusion of Theorem 3 holds.∎

Proof of Theorem 4: Theorem 4 alters the hypotheses of Theorem 3 in a manner which makes all comparison functions ψ_{ij}, $i=1,\ldots,\ell$, $j=1,2,3$ of the same order of magnitude. Composite system $\mathcal{S}$ may be shown to be exponentially stable if the comparison functions ψ_1, ψ_2 and ψ_3 in (148) and (152) are of the same order of magnitude. This is accomplished as follows.

By the definition given in Chapter II on notation regarding functions of the same order of magnitude, it follows from hypothesis (i) of Theorem 4 that there is a function $\psi \in KR$ (e.g., $\psi = \psi_{11}$), and positive constants k_{ij}, $i=1,\ldots,\ell$, $j=1,2,3$, such that

$$\psi_{i1}(r) \geq k_{i1}\psi(r) \;, \quad \psi_{i2}(r) \leq k_{i2}\psi(r) \;,$$

$$\psi_{i3}(r) \geq k_{i3}\psi(r) \tag{153}$$

for all $r \geq 0$, $i=1,\ldots,\ell$. Define the functions ψ_1, ψ_2 and ψ_3 as follows

$$\psi_1(r) = \min_i (\alpha_i k_{i1})\psi(r) \quad , \tag{154}$$

$$\psi_2(r) = (\sum_{i=1}^{\ell} \alpha_i k_{i2})\psi(r) \quad , \tag{155}$$

and

$$\psi_3(r) = -\lambda_M(S) \min_i (k_{i3})\psi(r) \quad , \tag{156}$$

each of class KR. Then one obtains, using the norm $\|\cdot\|$ as defined in (14) and the fact that ψ is a strictly increasing function,

$$\begin{aligned} \sum_{i=1}^{\ell} \alpha_i \psi_{i1}(\|a_i\|_i) &\geq \sum_{i=1}^{\ell} \alpha_i k_{i1} \psi(\|a_i\|_i) \\ &\geq \min_i (\alpha_i k_{i1}) \sum_{i=1}^{\ell} \psi(\|a_i\|_i) \\ &\geq \min_i (\alpha_i k_{i1}) \max_i \psi(\|a_i\|_i) \\ &= \min_i (\alpha_i k_{i1}) \psi(\max_i \|a_i\|_i) = \psi_1(\|a\|) \quad . \end{aligned} \tag{157}$$

Also,

$$\begin{aligned} \sum_{i=1}^{\ell} \alpha_i \psi_{i2}(\|a_i\|_i) &\leq \sum_{i=1}^{\ell} \alpha_i k_{i2} \psi(\|a_i\|_i) \\ &\leq (\sum_{i=1}^{\ell} \alpha_i k_{i2}) \max_i \psi(\|a_i\|_i) \end{aligned}$$

$$= \left(\sum_{i=1}^{\ell} \alpha_i k_{i2}\right)\psi(\max_i \|a_i\|_i)$$

$$= \psi_2(\|a\|) \quad . \tag{158}$$

Finally, since $\lambda_M(S) < 0$, one has

$$\lambda_M(S) \sum_{i=1}^{\ell} \psi_{i3}(\|a_i\|_i) \leq \lambda_M(S) \sum_{i=1}^{\ell} k_{i3}\psi(\|a_i\|_i)$$

$$\leq \lambda_M(S) \min_i (k_{i3}) \sum_{i=1}^{\ell} \psi(\|a_i\|_i)$$

$$\leq \lambda_M(S) \min_i (k_{i3}) \max_i \psi(\|a_i\|_i)$$

$$= \lambda_M(S) \min_i (k_{i3})\psi(\max_i \|a_i\|_i) = -\psi_3(\|a\|) \quad . \tag{159}$$

Therefore (146), (147) and (151) are satisfied, and since ψ_1, ψ_2 and ψ_3 are of the same order of magnitude, composite system $\mathcal{S}$ satisfies the conditions of Theorem 2. Hence, the conclusion of Theorem 4 holds.∎

Proof of Theorems 5 and 6: These two theorems follow from the definition of M-matrices and the remark preceding Theorem 5. Let D be the matrix defined in (20) and let $W = ((\delta_{ij}\alpha_j))$, $i,j=1,\ldots,\ell$. Then it is easily shown that the matrix $S = -\frac{1}{2}[WD + D'W]$ is identical to the test matrix S of Theorem 3. By the result cited from the theory of M-matrices (see Appendix C) one may conclude that if D is an M-matrix, then constants $\alpha_i > 0$, $i=1,\ldots,\ell$, exist such that $WD + D'W$ is positive definite, and hence S is negative definite. It is sufficient therefore to assume that D is an M-matrix in place of hypothesis (iii) of Theorem 3.

In Theorem 5 the test matrix is assumed to satisfy the condition on the off-diagonal elements and condition (i) of Definition 7. These conditions are given in place of hypothesis (iii) of Theorem 3 and since they imply that D is an M-matrix the conclusion of Theorem 5 holds.

In Theorem 6 condition (ii) of Definition 7 is used in place of condition (i). The positive vector required is $x = (\lambda_i)$, $i=1,\ldots,\ell$, and the condition $Dx > 0$ may be written as

$$- \lambda_i(\sigma_i + b_{ii}) - \sum_{j=1}^{\ell} \lambda_j b_{ij} > 0 \quad , \quad i=1,\ldots,\ell \quad . \tag{160}$$

A simple rearrangement of terms yields (21) and therefore the conclusion of Theorem 6 holds.

In an analogous manner conditions (iii), (iv) and (v) could be used to obtain additional results. Condition (v) could even be expressed in terms of the stability of the linear system $\dot{x} = - Dx$. Such extensions however are obvious once their possibility is pointed out.∎

<u>Proof of Theorem 7</u>: This theorem is a simple consequence of the remark (see Appendix C) preceding Theorem 7 which is another result from the theory of M-matrices. The modification suggested is simply insufficient to change D from its M-matrix condition. The conclusions of Theorems 5 and 6 are therefore unchanged. Hence the conclusion of Theorem 7 holds.∎

XI. APPENDIX B

The proofs of the main results of Chapter VI regarding the stability of large-scale stochastic systems will be given subsequently. First, however, some preliminary results are in order.

The following matrix inequality will be used

$$\mathrm{tr}[\sigma' B\sigma] = \sum_{i,j,k} \sigma_{ji} B_{jk} \sigma_{ki} = \sum_i \sigma'_{.i} B \sigma_{.i}$$

$$\leq e \sum_i |\sigma_{.i}|^2 = e \sum_{i,j} \sigma_{ij}^2 = e \|\sigma\|_m^2 \tag{161}$$

where $\sigma_{.i} = (\sigma_{ji})$ denotes the i^{th} column vector of σ, and e is a scalar such that $u'Bu \leq e|u|^2$.

Some forms of $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{G}$ for special cases will also be used. Consider first $\mathcal{L}V_i$ for system (85)

$$\mathcal{L}V_i(a_i) = m(a)' \nabla_a V_i(a_i) + \frac{1}{2} \mathrm{tr}[\sigma(a)' \nabla_{aa} V_i(a_i) \sigma(a)]$$

$$= \sum_{j=1}^{\ell} [m_j(a_j) + g_j(a)]' \nabla_{a_j} V_i(a_i)$$

$$+ \frac{1}{2} \sum_{j,k,m=1}^{\ell} \mathrm{tr}[\sigma_{kj}(a_j)' \nabla_{a_k a_m} V_i(a_i) \sigma_{mj}(a_j)]$$

$$= \sum_{j=1}^{\ell} [m_j(a_j) + g_j(a)]' \delta_{ij} \nabla_{a_i} V_i(a_i)$$

$$+ \frac{1}{2} \sum_{j,k,m=1}^{\ell} \mathrm{tr}[\sigma_{kj}(a_j)' \delta_{ki} \delta_{mi} \nabla_{a_i a_i} V_i(a_i) \sigma_{mj}(a_j)]$$

$$= [m_i(a_i) + g_i(a)]' \nabla_{a_i} V_i(a_i)$$

$$+ \frac{1}{2} \sum_{j=1}^{\ell} \mathrm{tr}[\sigma_{ij}(a_j)' \nabla_{a_i a_i} V_i(a_i) \sigma_{ij}(a_j)]$$

$$= \mathcal{L}_i V_i(a_i) + g_i(a)' \nabla_{a_i} V_i(a_i)$$

$$+ \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \mathrm{tr}[\sigma_{ij}(a_j)' \nabla_{a_i a_i} V_i(a_i) \sigma_{ij}(a_j)] \quad . \qquad (162)$$

If in (85), ζ_t^j, $j=1,\ldots,\ell$, are Poisson processes then let $p_i^j \, \Delta t + o(\Delta t)$ be the probability of a jump experienced by the i^{th} component of q_t^j in any interval of length Δt. Let the corresponding jump amplitude distribution be $P_i^j(dq_i^j)$. The infinitesimal generator for (85) may then be written as

$$\mathcal{D}\, V(a) = m(a)' \nabla_a V(a)$$

$$+ \sum_{j=1}^{\ell} \sum_{k=1}^{m_j} \int_{q_k^j} [V(a + \sigma_{\cdot j,k}(a_j) q_k^j) - V(a)] p_k^j P_k^j(dq_k^j)$$

(163)

where $\sigma_{\cdot j,k}$ denotes the k^{th} column of $\sigma_{\cdot j} = (\sigma_{ij})$.

Then if $\sigma_{ij}(a_j) = 0$, $i \neq j$, $i,j=1,\ldots,\ell$, one obtains

$$\mathcal{D} V_i(a_i) = \sum_{j=1}^{\ell} [m_j(a_j) + g_j(a)]' \nabla_{a_j} V_i(a_i)$$

$$+ \sum_{j=1}^{\ell} \sum_{k=1}^{m_j} \int_{q_k^j} [V_i(a_i + \sigma_{ij,k}(a_j) q_k^j) - V_i(a_i)] p_k^j P_k^j(dq_k^j)$$

$$= [m_i(a_i) + g_i(a)]' \nabla_{a_i} V_i(a_i)$$

$$+ \sum_{k=1}^{m_i} \int_{q_k^i} [V_i(a_i + \sigma_{ii,k}(a_i) q_k^i) - V_i(a_i)] p_k^i P_k^i(dq_k^i)$$

$$= \mathcal{D}_i V_i(a_i) + g_i(a)' \nabla_{a_i} V_i(a_i) \quad . \tag{164}$$

The equivalence of $\mathcal{L}$ and $\mathcal{D}$ for quadratic V functions is shown as follows. Using (78), one has

$$\mathcal{D}\, a'Pa = m(a)' \nabla_a (a'Pa)$$

$$+ \sum_{i=1}^{m} \int_{q_i} [(a + \sigma_{.i}(a) q_i)' P (a + \sigma_{.i}(a) q_i) - aPa]$$

$$\cdot\, p_i P_i(dq_i)$$

$$= 2m(a)'Pa + \sum_{i=1}^{m} [a'P\sigma_{.i}(a) + \sigma_{.i}(a)'Pa] p_i$$

$$\cdot \int_{q_i} q_i P_i(dq_i)$$

$$+ \sum_{i=1}^{m} \sigma_{.i}(a)'P\sigma_{.i}(a)p_i \int_{q_i} q_i^2 P_i(dq_i)$$

$$= 2m(a)'Pa + tr[\sigma(a)'P\sigma(a)] \qquad (165)$$

where the normalizing conditions (76) and (77) have been used. For (75) one obtains

$$\mathcal{L}a'Pa = m(a)'\nabla_a(a'Pa) + \frac{1}{2} tr[\sigma(a)' \nabla_{aa}(a'Pa)\sigma(a)]$$

$$= 2m(a)'Pa + tr[\sigma(a)'P\sigma(a)] \qquad (166)$$

which is the same as (165) and establishes the equivalence.

Finally, for system (87) let $p_{jk}^i \Delta t + o(\Delta t)$ be the probability of a jump in y_t^i from y_j^i to y_k^i during any interval of length Δt. Then for $y_t = (y_t^i)$, $i=1,\ldots,\ell$, with independent components, the probability of a jump from $y_j = (y_j^i)$, $i=1,\ldots,\ell$ to $y_k = (y_k^i)$, $i=1,\ldots,\ell$, in an interval of length Δt has a nonzero first order coefficient only for transitions involving a single component. Therefore, in a manner analogous to the above results one obtains

$$\mathcal{A} V_i(a_i,y_j^i) = [f_i(a_i,y_j^i) + g_i(a,y_j)]'\nabla_{a_i} V_i(a_i,y_j^i)$$

$$+ \sum_{\substack{k=1 \\ k\neq j}}^{N_i} p_{jk}^i[V_i(a_i,y_k^i) - V_i(a_i,y_j^i)]$$

$$= \mathcal{Q}_i V_i(a_i,y_j^i) + g_i(a,y_j)' \nabla_{a_i} V_i(a_i,y_j^i) \quad . \tag{167}$$

It is now possible to proceed with the proofs.

Proof of Theorem 10: By hypothesis (ii) of Theorem 10 and the definition of Property C there exist Lyapunov functionals V_i for each subsystem $\mathcal{S}_i$. Choose as a Lyapunov functional for the composite system $\mathcal{S}$ the weighted sum

$$V(a,\cdot) = \sum_{i=1}^{\ell} \alpha_i V_i(a_i,\cdot) \tag{168}$$

where the constants $\alpha_i > 0$, $i=1,\ldots,\ell$, are as yet undetermined. It follows from Property C that

$$\sum_{i=1}^{\ell} \alpha_i \psi_{i1}(|a_i|) \leq V(a,\cdot) \leq \sum_{i=1}^{\ell} \alpha_i \psi_{i2}(|a_i|) \quad . \tag{169}$$

Therefore, as in Appendix A, there exist functions ψ_1, ψ_2 ε KR such that

$$\psi_1(|a|) \leq V(a,\cdot) \leq \psi_2(|a|) \quad . \tag{170}$$

Note that because of hypothesis (i), which sets $\sigma_{ij} = 0$ for $i \neq j$, Equations 162, 164 and 167 each have the form

$$AV_i(a_i,\cdot) = A_i V_i(a_i,\cdot) + g_i(a,\cdot)' \nabla_{a_i} V_i(a_i,\cdot) \quad . \tag{171}$$

Therefore, for the cases of interest, it follows from the linearity of A and hypotheses (ii) and (iii) that

$$AV(a,.) = \sum_{i=1}^{\ell} \alpha_i AV_i(a_i,\cdot)$$

$$= \sum_{i=1}^{\ell} \alpha_i [A_i V_i(a_i,\cdot) + g_i(a,\cdot)' \nabla_{a_i} V_i(a_i,\cdot)]$$

$$\leq \sum_{i=1}^{\ell} \alpha_i \{ \sigma_i \psi_{i3}(|a_i|)$$

$$+ [\psi_{i3}(|a_i|)]^{1/2} \sum^{\ell} b_{ij} [\psi_{j3}(|a_j|)]^{1/2} \}$$

$$= u'Su \quad , \tag{172}$$

where $u = ([\psi_{i3}(|a_i|)]^{1/2})$, $i=1,\ldots,\ell$, and S is the test matrix of hypothesis (iv). It follows, as in the proof of Theorem 3 in Appendix A, that

$$AV(a,\cdot) \leq -\psi_3(|a|) \tag{173}$$

where $\psi_3 \in K$. All conditions of Theorem 8 have been satisfied for $\mathcal{S}$ and therefore the conclusion of Theorem 10 holds.∎

Proof of Theorem 11: Choose the Lyapunov functional

$$V(a) = \sum_{i=1}^{\ell} \alpha_i V_i(a_i) \tag{174}$$

as in (168). Then one obtains the analogous condition

$$\psi_1(|a|) \leq V(a) \leq \psi_2(|a|) \tag{175}$$

where ψ_1, $\psi_2 \varepsilon$ KR.

In addition, for this case one obtains the following result from hypotheses (ii)-(v) of Theorem 11, and Equations 161 and 162. That is,

$$\mathcal{L}V(a) = \sum_{i=1}^{\ell} \alpha_i \mathcal{L}V_i(a_i)$$

$$= \sum_{i=1}^{\ell} \alpha_i \{\mathcal{L}_i V_i(a_i) + g_i(a)' \nabla_{a_i} V_i(a_i)$$

$$+ \frac{1}{2} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \mathrm{tr}[\sigma_{ij}(a_j)' \nabla_{a_i a_i} V_i(a_i) \sigma_{ij}(a_j)]\}$$

$$\leq \sum_{i=1}^{\ell} \alpha_i \{\sigma_i \psi_{i3}(|a_i|)$$

$$+ [\psi_{i3}(|a_i|)]^{1/2} \sum_{j=1}^{\ell} b_{ij} [\psi_{j3}(|a_j|)]^{1/2}$$

$$+ \frac{1}{2} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} e_i \| \sigma_{ij}(a_j) \|_m^2 \}$$

$$\leq \sum_{i=1}^{\ell} \alpha_i (\sigma_i + b_{ii}) \{[\psi_{i3}(|a_i|)]^{1/2}\}^2$$

$$+ \sum_{\substack{i,j=1 \\ i\neq j}}^{\ell} \frac{1}{2}(\alpha_i b_{ij} + \alpha_j b_{ji}) [\psi_{i3}(|a_i|)]^{1/2} [\psi_{j3}(|a_j|)]^{1/2}$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \frac{1}{2} \alpha_i e_i d_{ij} \{[\psi_{j3}(|a_j|)]^{1/2}\}^2$$

$$= u'Su \tag{176}$$

where $u = ([\psi_{i3}(|a_i|)]^{1/2})$, $i=1,\ldots,\ell$, and S is the test matrix of hypothesis (vi). It follows as in the previous proof that

$$\mathscr{L}V(a) \leq -\psi_3(|a|)$$

where $\psi_3 \in K$, and therefore the conclusion of Theorem 11 holds.∎

Proof of Theorem 12: Since each subsystem $\mathscr{S}_i$ is now assumed to possess Property D, it is quite easily shown that the comparison functions ψ_1, ψ_2 and ψ_3 of Theorems 10 and 11 could be chosen as

$$\psi_1(r) = \min_i (\alpha_i c_{i1}) r^2 \quad , \tag{178}$$

$$\psi_2(r) = \max_i (\alpha_i c_{i2}) r^2 \quad , \tag{179}$$

and

$$\psi_3(r) = \lambda_M(S) r^2 \quad . \tag{180}$$

Then by Theorem 9, the conclusion of Theorem 12 holds.∎

Proof of Theorem 13: This result follows directly from Theorems 11 and 12 and the equivalence of $\mathcal{L}$ and $\mathcal{D}$, given the quadratic functionals V_i. The hypothesis analogous to hypothesis (iv) of Theorem 11 is unnecessary since it is satisfied automatically with $e_i = 2\lambda_M(P_i)$. In (95), $2P_i a_i$ has been substituted for $\nabla_{a_i} V_i(a_i)$. It follows therefore that the conclusion of Theorem 13 holds.∎

XII. APPENDIX C

In Chapter IV two properties of M-matrices were used to establish Theorems 5-7. These results are proven here.

<u>Theorem</u>: Let D be an M-matrix. Then there exists a diagonal matrix W with positive diagonal elements such that $WD + D'W$ is positive definite.∎

<u>Proof</u>: Let $P = WD + D'W$. Then since D is an M-matrix and W has a positive diagonal, it follows that $p_{ij} = w_i d_{ij} + w_j d_{ji} \leq 0$ for $i \neq j$. That is, P has nonpositive off-diagonal elements.

By conditions (ii) and (iii) of Definition 7 there exist vectors $x > 0$ and $y > 0$ such that $U \triangleq Dx > 0$ and $V \triangleq D'y > 0$. Now choose $w_i = y_i/x_i$, $i=1,\ldots,\ell$. Then

$$\begin{aligned} Px &= (WD + D'W)x = W(Dx) + D'y \\ &= WU + V > 0 \, . \end{aligned} \tag{181}$$

That is, P satisfies condition (ii) of Definition 7.

The matrix P is seen to be an M-matrix and therefore by condition (v) the eigenvalues of P have positive real parts. But, since P is symmetric, it follows that $P = WD + D'W$ is positive definite. This proves the theorem.∎

<u>Theorem</u>: Let D be an M-matrix and let $\mu < \min \mathrm{Re}[\lambda(D)]$. Then $D - \mu I$ is an M-matrix.∎

<u>Proof</u>: If $\lambda(D)$ is an eigenvalue of D, then

$$\det[\lambda(D)I-D] = \det[(\lambda(D)-\mu)I - (D-\mu I)] = 0 \quad . \qquad (182)$$

That is, $\lambda(D)-\mu$ is an eigenvalue of $D-\mu I$. Since μ is real, it follows that

$$\mathrm{Re}[\lambda(D)-\mu] = \mathrm{Re}[\lambda(D)]-\mu \geq \min \mathrm{Re}[\lambda(D)]-\mu > 0 \quad . \qquad (183)$$

The matrix $D-\mu I$ has nonpositive off-diagonal elements and by (183) the real parts of the eigenvalues of $D-\mu I$ are positive. Therefore by Definition 7, using condition (v), it follows that $D-\mu I$ is an M-matrix. This proves the theorem.∎